A quantum geometric model of similarity

Emmanuel M. Pothos[1], Jerome R. Busemeyer[2], & Jennifer S. Trueblood[3]

Affiliations, correspondence: 1: Department of Psychology, City University, London, EC1V 0HB, UK, emmanuel.pothos.1@city.ac.uk; 2: Department of Psychological and Brain Sciences, Indiana University, Bloomington 47468, Indiana, USA, jbusemey@indiana.edu; 3: Department of Cognitive Sciences, 3151 Social Sciences Plaza, University of California, Irvine, Irvine, CA 92697-5100, USA, jstruebl@uci.edu

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Abstract

No other study has had as great an impact on the development of the similarity literature as that of Tversky (1977), which provided compelling demonstrations against all the fundamental assumptions of the popular, and extensively employed, geometric similarity models. Notably, similarity judgments were shown to violate symmetry and the triangle inequality, and also be subject to context effects, so that the same pair of items would be rated differently, depending on the presence of other items. Quantum theory provides a generalized geometric approach to similarity and can address several of Tversky's (1997) main findings. Similarity is modeled as quantum probability, so that asymmetries emerge as order effects, and the triangle equality violations and the diagnosticity effect can be related to the context-dependent properties of quantum probability. We so demonstrate the promise of the quantum approach for similarity and discuss the implications for representation theory in general.

Keywords: similarity, metric axioms, symmetry, triangle inequality, diagnosticity, quantum probability

I. Introduction

The notion of similarity is, in equal measure, a famous hero and a notorious villain in psychology. Across most areas of psychology, similarity plays a fundamental role (e.g., Goldstone, 1994; Pothos, 2005; Sloman & Rips, 1998), but equally its various formalizations have been the source of much criticism and debate (e.g., Goodman, 1972). A popular approach to similarity is a geometric one, according to which stimuli/ exemplars/ concepts are represented as points in a multidimensional psychological space, with similarity being a function of distance in that space. This geometric approach is exemplified in Shepard's (1987) famous law of generalization, according to which similarity is an exponentially decaying function of distance, and is heavily used in influential cognitive models of categorization, such as exemplar and prototype theory. It is fair to say that cognitive psychology cannot resist using a geometric approach to similarity.

This reliance on the geometric approach to similarity is surprising because it has been subject to intense, and, in some cases, highly compelling criticisms. The most complete and impactful expression of this criticism is that of Tversky (1977). Tversky's work has had a profound influence on the development of the similarity literature (over 2,200 citations), partly because his objections to geometric similarity models concern the most basic properties of such models – the metric axioms, that is, the fundamental properties that any similarity measure based on distance must obey. Thus, if the metric axioms are shown to be inconsistent with psychological similarity, then any distance model of similarity is essentially incorrect. Tversky's (1977) demonstration is a rare one, in that he has been able to convincingly argue against an entire modeling framework, rather than particular models. This is because his arguments were not dependent on e.g. particular parametric configurations, rather they concerned the fundamental properties of *any* model of similarity based on distance in psychological space (though see Nosofsky, 1991, for a parametric way to produce an asymmetric distance-based similarity measure). It is not surprising that Tversky's (1977) demonstrations have come to be accepted as the golden standard of key results any successful similarity model should cover (Ashby & Perrin, 1988; Bowdle & Gentner, 1997; Goldstone & Son, 2005; Krumhansl, 1978).

In brief, Tversky (1977) showed that similarity judgments violate minimality (identical objects are not always judged to be maximally similar), symmetry (the similarity of A to B can be different from that of B to A), and the triangle inequality (the distance between two points is always shorter directly, than via a third point). Moreover, he showed that the similarity between the same two objects can be affected by which other objects are present (called the diagnosticity effect). In the typical tradition of his work, part of the reason why his findings have had the influence they did is because they go against basic logic. For example, concerning his most famous result, violations of symmetry, if similarity is determined by distance, then how could it be the case that the similarity/ distance between two objects depends on the order in which they are considered? Yet, when he asked participants to choose between the statements 'China is similar to Korea' vs. 'Korea is similar to China' (actually North Korea and Red China, but for simplicity we will just talk about Korea and China'), 66 out of 69 participants selected the latter statement as more agreeable, implying that the similarity of Korea to China (denoted as S*im*(Korea, China)) is higher than that of China to Korea (denoted as *Sim* (China, Korea)). Thus, this result provided a compelling (and *retrospectively* intuitive) violation of symmetry in similarity. Tversky employed several other pairs of countries, stimuli from other domains, and alternative procedures (see also Bowdle & Gentner, 1997, Catrambone, Beike, & Niedenthal, 1996, Holyoak & Gordon, 1983, Op de Beeck, Wagemans, & Vogels, 2003, Ortony et al., 1985, and Rosch, 1975). Note that some researchers have questioned the reality of asymmetries in similarity. For example, Gleitman et al. (1996) suggested that in directional similarity statements, we cannot assume that, e.g., Korea gives rise to the same representation in the target position, as it does in the referent position. But Gleitman et al.'s (1996) analysis cannot explain why it is more intuitive to place, e.g., Korea in the referent, as opposed to the target, position, the absence of asymmetries in some cases (Aguilar & Medin, 1999), and the demonstration of similarity asymmetries with non-linguistic measures (Hodgetts & Hahn, 2012).

We will present what can be labeled a *quantum similarity* model. Quantum probability (QP) theory is a theory for how to assign probabilities to events (for more refined characterizations see e.g. Aerts & Gabora, 2005; Atmanspacher, Romer, & Wallach, 2006; Busemeyer & Bruza, 2012; Khrennikov, 2010). QP theory is a geometric theory of probability. It is analogous to classical probability theory, though QP theory and classical theory are founded from different sets of axioms (the Kolmogorov and Dirac/ von Neumann axioms respectively) and so are subject to alternative constraints. QP theory is based on linear algebra, augmented with a range of assumptions and theorems (such as the Kochen-Specker theorem and Gleason's theorem; Busemeyer & Bruza, 2012; Hughes, 1989; Isham, 1989; Khrennikov, 2010). Note that a quantum approach to cognitive modeling does not introduce assumptions regarding neural implementation and we are agnostic on this issue. Specifically, operations which are quantum-like can emerge at the computational level from a classical brain (Atmanspacher & beim Graben, 2007) and do not assume quantum neural computations (this latter thesis is very controversial; Hameroff, 2007; Litt et al., 2006).

A unique feature of the quantum similarity model is that, whereas previous models would equate objects with individual points or distributions of points, in the quantum model, objects are entire subspaces of potentially very high dimensionality. This is an important generalization of geometric models of similarity, as it leads to a naturally asymmetric similarity measure.

The quantum similarity model follows the recent interest in the application of quantum probability (QP) theory to cognitive modeling. Applications of QP theory have been presented in decision making (Blutner et al., in press; Busemeyer, Wang, & Townsend, 2006; Busemeyer et al., 2011; Bordley, 1998; Lambert-Mogiliansky, Zamir, & Zwirn, 2009; Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011; Wang & Busemeyer, in press; Yukalov & Sornette, 2010), conceptual combination (Aerts, 2009; Aerts & Gabora, 2005; Blutner, 2008; Bruza et al., under review), memory (Bruza, 2010; Bruza et al., 2009), and perception (Atmanspacher, Filk, & Romer, 2004). Psychological models based on quantum probability seem to work well (for overviews see Busemeyer & Bruza, 2009; Bruza et al., 2009; Khrennikov, 2004; Pothos & Busemeyer, in press) and add to the increasing realization that the application of QP need not be restricted to physics. For example, QP has also been applied to areas as diverse as economics (Baaquie, 2004) and information theory (Nielsen & Chuang, 2010).

We first present QP theory and motivate our similarity model. Subsequently, we consider three main results from Tversky (1977). Violations of symmetry, violations of the triangle inequality, and the diagnosticity effect. Violations of symmetry provide and most compelling and intuitive evidence against (simple) geometric representational models. Moreover, the diagnosticity effect is obviously impossible to reconcile with similarity models based on distance alone, as it shows that similarity judgments between the same two elements might be affected by the presence of other elements. Note, we do not consider violations of minimality, i.e., the finding that naïve observers do not always assign the maximum similarity rating for pairs of identical stimuli. Violations of minimality can simply be explained by noise in the system, so that the same stimulus presented twice would lead to slightly different internal representations. Violations of minimality have been typically demonstrated in confusability experiments, whereby participants have to decide whether two consecutively presented stimuli are identical or not. But, lack of identity judgments for identical stimuli can be explained if the time course of sampling stimulus information is stochastic and, moreover, it is straightforward to couple minimality violations and stimulus complexity (Lamberts, 2000; Nosofsky & Palmeri, 1997).

II. QP theory and geometric similarity

Representation in QP theory is based on a multidimensional space, in which different subspaces correspond to different entities. The current state of the system is described by a vector in this space (the knowledge state vector). Projecting the state onto different subspaces and computing the squared length of the projected vector tells us about the consistency between the state vector and these other entities in the quantum space. We present the quantum model in three steps. First, we outline the relevant elements of quantum theory. Second, we discuss the assumptions for how operations in quantum theory can be employed to provide a model of psychological similarity. Third, we briefly consider some prior general motivating considerations and criticisms. Finally, each empirical situation we consider involves some specific information. In each corresponding section, we discuss how this information can be incorporated in the quantum model.

Main elements of quantum theory

We require a psychologically realistic quantum space, which represents all the knowledge of a person. Therefore, this knowledge space can have a very high dimensionality (potentially infinite [Footnote 1]). The state vector, ψ, is a unit length vector in the knowledge space; we will refer to ψ as the current knowledge state vector or just the state vector. It corresponds, broadly speaking, to whatever a person is thinking at a particular time (e.g., a knowledge state could be determined by the experimental instructions). If we employ Dirac notation, then $|\psi\rangle$ corresponds to a column vector and $\langle\psi|$ corresponds to a row version of this vector. The expression $\langle\psi|\square\rangle$ indicates a standard dot or inner product between vectors $|\psi\rangle$ and $|\square\rangle$.

Subspaces of the knowledge space represent different concepts, like China. A subspace could be a ray spanned by a single vector, or a plane spanned by a pair of vectors, or a three dimensional space spanned by three vectors, etc. Suppose that the China subspace is spanned by two orthonormal vectors, $|v_1\rangle$ and $|v_2\rangle$ (that is, the China subspace is two-dimensional; we will shortly consider how meaning may be ascribed to $|v_1\rangle$, $|v_2\rangle$). That is, $|v_1\rangle$ and $|v_2\rangle$ are basis vectors for the China subspace. Then, the concept of China is basically *all* the vectors of the form $a \cdot |v_1\rangle + b \cdot |v_2\rangle$, where $a^2 + b^2 = 1$ (as is required for a state vector in quantum theory). Note that this statement is different from, though obviously related to, the statement that a category corresponds to a region of psychological space (Ashby & Perrin, 1988; Gärdenfors, 2000; Nosofsky, 1984). So, to represent China with a subspace is to assume that the concept China is the collection of all thoughts, $a|v1\rangle + b|v2\rangle$, which are consistent with this concept. For example, our knowledge of China would include information about culture, food, language etc. The representation of China as a subspace implies that all these properties have to be contained in the China subspace. Therefore, the greater the range of thoughts we can have about a concept (e.g., properties or statements), the greater the dimensionality of the subspace. If we represent China as a two dimensional subspace and Korea as a one dimensional subspace, this means that we can have a greater range of thoughts for China, than for Korea, which is equivalent to assuming that we have greater knowledge for China than for Korea.

The representation of China as a subspace is consistent with the idea that properties are not uniquely chained to particular concepts. For example, suppose that my current thought is $|\psi\rangle = |Chinese\ \rangle$ (I am thinking about the Chinese language). Then, this thought would be included in the China subspace, but it would also be included in many other subspaces (for example, the subspace corresponding to my concept of 'foreign languages'). That is, a particular thought can be included in several subspaces at the same time.

Note that a thought of the form $|\psi\rangle = a|v1\rangle + b|v2\rangle$ is neither about $|v1\rangle$ nor $|v2\rangle$, but rather reflects the *potentiality* that the person will end up definitely thinking about $|v1\rangle$ or $|v2\rangle$. For example, if $|a|>|b|$, then this means that the person has a greater potential to think of $|v1\rangle$ than $|v2\rangle$. In QP theory, states like $a|v1\rangle + b|v2\rangle$ are called superposition states and the fact that we cannot ascribe definite meaning to such states is the result of a famous theorem (the Kochen-Specker theorem).

We next further consider the meaning of vectors $|v_1\rangle$ and $|v_2\rangle$, in the claim that they span the China subspace. We could consider each such vector as a separate, distinct property of China. However, in general, different subsets of properties of a particular concept are likely to correlate with each other. For example, the properties relating to Chinese food are likely to correlate with properties relating to the general health of the average Chinese person. We so interpret $|v_1\rangle$ and $|v_2\rangle$ as vectors which correlate with sets of properties, which are characteristic of China. How to determine the set of appropriate vectors, properties, or dimensions is an issue common to all geometric approaches to similarity. Recent work, especially by Storms and collaborators (e.g., De Deyne et al., 2008), shows that this challenge can be overcome, for example, through the collection of similarity information across several concepts or feature elicitation. Then, the relatedness of the properties will determine the overall dimensionality of the concept.

Given the China subspace and a state vector, we can examine the degree to which the state vector is consistent with the subspace, by projecting it onto the subspace. In quantum theory, this operation is achieved by a projector. A projector can be represented by a matrix, which takes a vector and projects it (lays it down) onto a particular subspace. For example, say P_{China} and P_{Korea} are the projectors on the China and Korea subspaces, respectively. The projection $P_{Korea} \cdot |\psi\rangle$ represents the match between the current knowledge state and Korea, in other words, it computes the part of the vector $|\psi\rangle$ which is restricted or contained in the Korea subspace. More specifically, suppose that $|Korea\rangle$ is a normalized vector which is used to represent our knowledge of Korea. Then, the projector onto the one-dimensional subspace or ray spanned by $|Korea\rangle$ is denoted by $P_{Korea} = |Korea\rangle\langle Korea|$ (this corresponds to the matrix formed by the outer product of the column vector and the row vector).

In Figure 1, for example,, we are projecting vector B onto vector A. Let us assume that both vectors are unit length. Then, the projection (indicated with the thick yellow line) would be another vector, specifically, the part of B which is contained in A. This is given by $|A\rangle\langle A|B\rangle$, noting that $P_A = |A\rangle\langle A|$ is the projector onto the *A* ray. Indeed, the notation $|A\rangle\langle A|B\rangle$ indicates a multiplication between a vector $|A\rangle$ and an inner product $\langle A|B\rangle$. But, from elementary geometry, we have that the inner product between two real vectors is $\langle A|B\rangle = |A| \cdot |B| \cdot cos\theta$, where θ is the angle between the two vectors (see also Sloman, 1993). If the two vectors are normalized, then $\langle A|B\rangle = cos\theta$.

-------------------FIGURE 1 ABOUT HERE--------------------

Furthermore, if we assume that the China concept is represented by a subspace spanned by vectors $|v1\rangle$ and $|v2\rangle$, then the mathematical expression for China is a projector denoted as $P_{China} = |v1\rangle\langle v1| + |v2\rangle\langle v2|$. This seems an elegant way to express our intuition that the representation of a concept corresponds to all the possible thoughts we can have about the concept. Thus, following from the example above, if we think about the Chinese language, then $|\psi\rangle = |Chinese\rangle$, and $P_{china}|Chinese\rangle = |Chinese\rangle$, showing that this is a thought included in the China concept (but, the China concept would include many other thoughts; e.g., $P_{china}|Chinese\ food\rangle = |Chinese\ food\rangle$). More generally the range of thoughts $|\psi\rangle$ such that $P_{China}|\psi\rangle = |\psi\rangle$ is the range of thoughts consistent with the concept of China or, equivalently, the thoughts which are part of the concept of China. Equally, $P_{foreign\ languages}|Chinese\rangle = |Chinese\rangle$, illustrating that this particular thought would be consistent with other concepts too.

One of the fundamental axioms of QP theory concerns how to derive a numerical measure of consistency between a subspace and a state vector, from the projected vector. Specifically, the length of the projection squared can be shown to be the *probability* that the state vector is consistent with the corresponding subspace. For example, the probability that a thought ψ is consistent with the China concept equals $||P_{China} \cdot |\psi\rangle||^2 = \langle\psi|P_{China}\psi\rangle$; for simplicity, we will denote the length of a vector A, $||A||$, as $|A|$. If the state vector is orthogonal to a subspace, then the probability is 0.

Throughout this section, projection was described as an operation revealing the consistency between a state vector and a subspace. In the next section, we suggest that, when the state vector, subspaces etc. are endowed with psychological meaning, this consistency can lead to a similarity measure.

Towards a quantum similarity model

We propose that the similarity between two concepts is determined by the sequential projection from the subspace corresponding to the first concept to the one for the second concept. Then, it can be shown that the similarity comparison is a process of thinking about the first of the compared concepts, followed by the second. Similarity in the quantum model is about how easy it is to think about one concept, from the perspective of another. Next, when a naïve observer is asked to rate the similarity between two concepts, she is typically not influenced by any prior thoughts. Therefore, we suggest that, prior to a similarity comparison, the state vector is set so that it does not bias the similarity judgment in favor of either concept. But, sometimes a similarity comparison is carried out in a way that reflects the influence of other concepts or stimuli. We suggest that, in some circumstances, it is appropriate to model this contextual influence by prior corresponding projections of the state vector. All these ideas can be implemented in a straightforward way.

Suppose we are interested in how similar Korea is to China. We have to project a neutral (as above) state vector to the subspace for one country and then to the subspace for the other. When there is no particular directionality in a sequential projection, we can either average the result from both directionalities or determine the directionality in another way (Busemeyer et al., 2011). However, similarity judgments are often formulated in a directional way (Tversky, 1977). When this is the case, we suggest that the directionality of the similarity judgment determines the directionality of the sequential projection, i.e., the syntax of the similarity judgment matches the syntax of the quantum computation. For example, the similarity of Korea to China would involve a process of thinking about Korea (subject, mentioned first) and then China (object, mentioned second), which corresponds to $sim(Korea, China) = |P_{China} \cdot P_{Korea} \cdot |\psi\rangle|^2$.

The link in quantum theory between projection and probability theory can help justify the use of $|P_{China} \cdot P_{Korea} \cdot |\psi\rangle|^2$ in computing sequential projection. Suppose the initial state is $|\psi\rangle$. From this initial state, the probability of a match to Korea equals $|P_{Korea} \cdot |\psi\rangle|^2$. If the person thinks that the current state matches the Korea subspace, then the new state is revised to become the normalized projection of the previous state onto the Korean subspace, so that $|\psi_{Korea}\rangle = \frac{P_{Korea} \cdot |\psi\rangle}{|P_{Korea} \cdot |\psi\rangle|}$. Finally, the probability that this conditional state is consistent with China equals $|P_{China} \cdot |\psi_{Korea}\rangle|^2$. Thus, $|P_{China} \cdot |\psi_{Korea}\rangle|^2 |P_{Korea} \cdot |\psi\rangle|^2$ exactly computes the sequence of probabilities for whether $|\psi\rangle$ is consistent with the Korea subspace and whether the (normalized) projection of $|\psi\rangle$ onto Korea is consistent with the China subspace. The product rule then follows from ,

$$|P_{China} \cdot |\psi_{Korea}\rangle|^2 |P_{Korea} \cdot |\psi\rangle|^2 = \left|P_{China} \cdot \frac{P_{Korea} \cdot |\psi\rangle}{|P_{Korea} \cdot |\psi\rangle|}\right|^2 |P_{Korea} \cdot |\psi\rangle|^2 = |P_{China} \cdot P_{Korea} \cdot |\psi\rangle|^2$$

(Busemeyer et al. , 2011).

As noted, in the absence of priming manipulation or contextual influence, we require the state vector to be neutral between the compared concepts, so that, in the China, Korea example, $|P_{Korea} \cdot |\psi\rangle|^2 = |P_{China} \cdot |\psi\rangle|^2$ (Appendix 1). Such an assumption is equivalent to that of a uniform prior in a Bayesian model (Trueblood & Busemeyer, 2012). Then, it is straightforward to show that $Sim(Korea, China) \propto |P_{China} \cdot |\psi_{Korea}\rangle|^2$, whereby the vector $|\psi_{Korea}\rangle$ is a normalized vector contained in the Korea subspace. Therefore, the quantity $|P_{China} \cdot |\psi_{Korea}\rangle|^2$ depends on only two factors, the geometric relation between the China and the Korea subspaces and, as we shall see, the relative dimensionality of the subspaces. That is, the outcome of the similarity comparison between China and Korea depends only on the relation of what we know about China and Korea, which seems appropriate, in the absence of prior priming. If one imagines a set of concepts all represented as rays, then rays closer to each other (smaller angles) indicate higher similarities.

Modifying the basic similarity calculation to take into account context was motivated from Tversky's (1977) diagnosticity effect, one of the most compelling demonstrations in the similarity literature. In his experiment, participants had to identify the country most similar to a particular target, from a set of alternatives, and the empirical results showed that pairwise comparisons were influenced by the available alternatives. Such an influence can be accommodated within the quantum similarity model.

Occasionally, what a person is thinking just prior to a comparison cannot be assumed to be irrelevant to the comparison. Suppose that the similarity of *A* and *B* is computed in a way that has to take into account the influence of some contextual information, *C*, which is represented by a particular subspace. This information *C* could correspond to the alternatives in Tversky's (1977) diagnosticity task. The similarity between *A* and *B* should then be computed as $Sim(A, B) = |P_B P_A |\psi'\rangle|^2 = |P_B |\psi'_A\rangle|^2 |P_A |\psi'\rangle|^2$, where $|\psi'\rangle = |\psi_C\rangle = \frac{P_C|\psi\rangle}{|P_C|\psi\rangle|}$ is no longer a state vector neutral between *A* and *B*, but rather one which reflects the influence of information *C*. If we minimally assume that the nature of this contextual influence is to think of *C*, prior to comparing *A* and *B*, then $Sim(A, B) = |P_B P_A |\psi'\rangle|^2 = |P_B P_A \frac{P_C|\psi\rangle}{|P_C|\psi\rangle|}|^2 = |P_B P_A P_C |\psi\rangle|^2 / |P_C |\psi\rangle|^2$. In other words, if the similarity comparison between *A* and *B* involves first thinking about *A* and then about *B*, then the same similarity comparison, in the context of some other information, *C* should involve an additional first step of thinking about *C*. Additional contextual elements correspond to further prior projections, though note that eventually this process must break down (there must be a limit to how many proximal items can impact on a decision).

As before, the link with probability justifies the choice of $|P_B P_A P_C|\psi\rangle|^2$, since $|P_B P_A P_C|\psi\rangle|^2 = |P_B P_A|\psi_C\rangle|^2 |P_C|\psi\rangle|^2 = |P_B|\psi_{AC}\rangle|^2 |P_A|\psi_C\rangle|^2 |P_C|\psi\rangle|^2$, where $|\psi_C\rangle = \frac{P_C|\psi\rangle}{|P_C|\psi\rangle|}$ and $|\psi_{AC}\rangle = \frac{P_A|\psi_C\rangle}{|P_A|\psi_C\rangle|}$. Therefore, the similarity comparison between *A* and *B* is now computed in relation to a vector which is no longer neutral, but contained within the *C* subspace. Depending on the relation between subspace *C* and subspaces *A* and *B*, contextual information can have a profound impact on a similarity judgment. Also, the term $|P_C|\psi\rangle|^2$ affects the overall magnitude of the similarity comparison, but we assume that a computation like $|P_B P_A P_C|\psi\rangle|^2$ can lead to a sense of similarity in relation to other, matched computations. Such an assumption follows from discussions on the flexibility of similarity response scales, e.g., depending on the range of available stimuli (Parducci, 1965).

Some prior justifications and criticisms for the quantum similarity model

Why is it valuable to explore the quantum similarity model? In feature-based representation approaches (Tversky, 1977), there is a mechanism for modeling differences in the extent of knowledge we have for different concepts. This is an obvious requirement for representation models since, clearly, for some concepts we have more knowledge than for others. But, a corresponding mechanism does not exist in classical geometric representation schemes, according to which a concept (or exemplar etc.) is represented by a single vector, such as $C = x_1 + x_2 + x_3 + \cdots + x_k$ (any coefficients have been absorbed in the vector). Classically, if one dimension is, e.g., length and another height, a point in the corresponding space is about a stimulus of a certain length and height. Crucially, *all* other represented stimuli will also have a representation in terms of the same two dimensions, length and height, and it is not possible to have stimuli in the same space, represented with a different set of features. The quantum model is a major departure from classical geometric representation schemes, in that it provides a rigorous framework for associating concepts with subspaces. As the dimensionality of subspaces can vary arbitrarily, there are no constraints in the number of features that can be employed in a representation.

Many researchers have pointed out a potential link between decision-making and similarity (Medin, Goldstone, & Markman, 1995). For example, in Tversky and Kahneman's (1983) famous experiment, participants rated as more probable the statement 'Linda is a bank teller and a feminist' than 'Linda is a bank teller', even though classically a conjunction can never be more probable than an individual statement. According to Tversky and Kahneman, this decision involves a process of similarity (Linda is more similar or representative of a feminist), rather than a process based on probabilistic inference. This idea was systematically explored in Shafir, Smith, and Osherson (1990), who reported that a measure of probability of whether a conjunctive statement applied to a person predicted a measure of membership of the person to the corresponding category. Is there a tension between decision-making as probabilistic inference vs. as a similarity process? Not if one adopts a quantum approach. In this work, $Sim(A, B) = |P_B P_A|\psi\rangle|^2$. In the quantum decision-making model for the conjunction fallacy of Busemeyer et al. (2011), $Prob(A \ \wedge then\ B) = |P_B P_A|\psi\rangle|^2$. The only difference between the two models concerns an assumption that probabilistic judgments are typically directionless, so that directionality has to be determined in another way. Thus, quantum theory shows how nearly identical computations can model both similarity and decision making judgments. This equivalence predicts that similarity findings (such as the diagnosticity effect) may well have analogues in decision-making situations (cf. Roe et al., 2001).

Specific aspects of the present proposal can be related to prior proposals. The squared distance between two unit length vectors *X*, *Y* in a psychological space is given by $||X - Y||^2 = |X|^2 + |Y|^2 - 2\langle X|Y\rangle = 2 - 2\langle X|Y\rangle$. Thus, the computation $|P_{China}|\psi_{Korea}\rangle|^2$ depends on the distance between the corresponding vectors, as in standard geometric model of equating dissimilarity with distance. Also, the idea that concepts are subspaces is related to the influential exemplar models of categorization (Nosofsky, 1984), in which a category is associated with a region in psychological space.

Sloman (1993) suggested that the similarity between two categories, *A* and *B*, can be computed as $(A, B) = \frac{F(A)\cdot F(B)}{|F(A)|\,|F(B)|}$, where *F*(*A*) and *F*(*B*) are the vectors representing the categories, the numerator is a dot product, and $|F(A)| = \langle A|A\rangle^{1/2}$. Our proposal and Sloman's similarity measure are identical, if one employs normalized vectors, and in the special case where subspaces are unidimensional rays. This can be easily seen by noting that, according to our proposal, in the case of unidimensional subspaces, $Sim(A, B) = |P_B P_A|\psi\rangle|^2 = ||B\rangle\langle B|A\rangle\langle A|\psi\rangle|^2$. But, by assumption, $|\langle B|\psi\rangle|^2 = |\langle A|\psi\rangle|^2$, so that we can write $Sim(A, B) \propto ||A\rangle\langle A|B\rangle|^2 = |\langle A|B\rangle|^2 \propto Sim(B, A)$. Sloman notes that his similarity measure is symmetric and the same applies to our measure, in this unidimensional case. It is noteworthy that Sloman's intuition effectively led him to some of the same constructs as those in the formal framework of QP theory.

We next consider potential criticisms. First, in the basic definition of similarity, how robust is the postulated sequence of projections? We proposed that $Sim(A, B) = |P_B \cdot P_A \cdot |\psi\rangle|^2$ but could one equally postulate that $Sim(A, B) = |P_A \cdot P_B \cdot |\psi\rangle|^2$? The answer is *mostly* no. Assuming that order of projection is matched to syntactic order follows from the interpretation of the state vector as current thought, so that $Sim(A, B) = |P_B \cdot P_A \cdot |\psi\rangle|^2$ corresponds to thinking about *A* first (because *A* is mentioned first) and then thinking about *B*. One could argue that projection sequence should be instead set by subject, predicate relations. But, we suggest that this is a more involved assumption, than syntactic order (e.g., what determines subject, predicate status or why should this information be relevant in similarity judgments?). The point is not that alternative assumptions for projection sequence cannot be made; rather, that an assumption based on syntactic order is fairly minimal. Relatedly, Tversky's (1977; see also Krumhansl, 1978) similarity model cannot produce asymmetric similarities, without assuming that the target or the referent has a higher salience in a comparison. The quantum model does not require such an assumption. We think that to require an assumption of higher salience is a more involved assumption, than one relating to just the order of consideration of the two predicates.

Second is the assumption that the initial state vector is neutral, e.g., $|P_A \cdot |\psi\rangle|^2 = |P_B \cdot |\psi\rangle|^2$, problematic? We think not because this assumption implies that $Sim(Korea, China) \propto |P_{China} \cdot |\psi_{Korea}\rangle|^2$, a quantity which depends only on the geometric relation between the Korea and the China subspaces and their relative dimensionality. This is what we require for a similarity judgment, in the absence of relevant context or priming. The initial state vector needs to be *something*. The assumption that the state vector is neutral with respect to the two subspaces, i.e., that $|P_A \cdot |\psi\rangle|^2 = |P_B \cdot |\psi\rangle|^2$, simply means that the state vector does not bias the similarity comparison.

III. Asymmetry, Korea-China

The prototypical experimental finding we are interested in is that $Sim(Korea, China) > Sim(China, Korea)$. Tversky (1977) explained this result by assuming that typical participants have more knowledge for China, than for Korea, so that China is more salient than Korea. Of these intuitions, the one that can be implemented in the quantum model relates to knowledge, since more extensive knowledge for a concept translates to a greater dimensionality for the corresponding subspace. We can write $Sim(Korea, China) = |P_{China} \cdot P_{Korea} \cdot |\psi\rangle|^2 = |P_{China} \cdot \psi_{Korea}|^2 |P_{Korea} \cdot |\psi\rangle|^2$ and, likewise, $Sim(China, Korea) = |P_{Korea} \cdot |\psi_{China}\rangle|^2 |P_{China} \cdot |\psi\rangle|^2$. As the initial state vector is neutral with respect to the China, Korea subspaces, we need to prove that $|P_{China}|\psi_{Korea}\rangle|^2 > |P_{Korea}|\psi_{China}\rangle|^2$, assuming that the dimensionality of the China subspace is greater than that of the Korea one ($|\psi_{Korea}\rangle$ and $|\psi_{China}\rangle$ are normalized projections within the respective subspaces).

Note first that for $|P_{China} \cdot |\psi_{Korea}\rangle|^2 \neq |P_{Korea} \cdot |\psi_{China}\rangle|^2$ it has to be the case that $P_{China} \cdot P_{Korea} \neq P_{Korea} \cdot P_{China}$. Two operators will not commute if at least some of the basis vectors of the corresponding subspaces are at oblique angles, which means neither identical nor opposite (orthogonal). Regarding the Korea-China example, we can safely assume that the Korea features will not all be either identical or opposite to the China ones, so that $P_{China} \cdot P_{Korea} \neq P_{Korea} \cdot P_{China}$.

In Appendix 2, with some simplifying assumptions, we prove that for a random two-dimensional subspace corresponding to China and a random ray corresponding to Korea, it will always be the case that $|P_{China}|\psi_{Korea}\rangle|^2 \geq |P_{Korea}|\psi_{China}\rangle|^2$. Note that deviance from an 100% prediction can be predicted by assuming that not all naïve observers know more about China than Korea. The intuition for why the quantum model works is that projection to a subspace of larger dimensionality will generally preserve more of the amplitude of the original vector, compared to projection to a subspace of smaller dimensionality. For example, if we have a vector $|k\rangle$ and a projector $P = |x\rangle\langle x| + |y\rangle\langle y|$, then, $P|k\rangle = |x\rangle\langle x|k\rangle + |y\rangle\langle y|k\rangle$, so that the amplitude of the projection would depend on the absolute magnitude of both $\langle x|k\rangle$ and $\langle y|k\rangle$. By contrast, in the projection $|x\rangle\langle x|k\rangle$ the amplitude of the projection depends just on $\langle x|k\rangle$. The larger the subspace, the more the 'opportunity' that the resulting projection will be large. For example, in Figure 2, in both panels the green line corresponds to a one-dimensional subspace (e.g., Korea), the yellow plane to a two-dimensional subspace (e.g., China), and the black solid line halfway the two to the state vector. The length of the first projection corresponds to a solid blue line and, by assumption, is the same regardless of whether we project to the ray or onto the plane. But, the length of the second projection (the solid red line) differs depending on whether it is to a ray or to a plane, so that when this second projection is onto the plane, it is longer.

We next explored the quantum model prediction computationally. Consider a one-dimensional Korea subspace and a two-dimensional China one, which reside in a three-dimensional space. We make no assumptions regarding the relation between the Korea, China subspaces. Thus, the projector for the Korea subspace was the outer product $P_{Korea} = |korea\rangle\langle korea|$, where $|korea\rangle$ was a random vector (note all vectors are normalized). For China, we need to specify two orthonormal basis vectors, call them $|vector1\rangle$ and $|vector2\rangle$ [Footnote 2]. The $|vector1\rangle$ was another random vector. We next created another random vector, $|Random\rangle$, and computed $|vector2\rangle$ as the normalized $(I - |vector1\rangle\langle vector1|) \cdot |Random\rangle$, where *I* is the three-dimensional identity matrix. The way this works is because the projector to the orthogonal complement of a subspace W is given by $P_{W^\perp} = I - P_W$). Therefore, $(I - |vector1\rangle\langle vector1|) \cdot |Random\rangle$ is the part of $|Random\rangle$ which is orthogonal to $|vector1\rangle$. Thus,

$P_{china} = |vector1\rangle\langle vector1| + |vector2\rangle\langle vector2|$. The state vector $|\psi\rangle$ was computed so that $|P_{korea} \cdot |\psi\rangle|^2 = |P_{china} \cdot |\psi\rangle|^2$, as in Appendix 1. With 100,000 repetitions of this scheme, in 100% of all cases $Sim(Korea, China) > Sim(China, Korea)$, as required.

We further illustrate the model in a number of ways. First, we ran a variation such that China was a random ray (like Korea), instead of a random plane. As the projections to both Korea and China are to rays, we expect no difference between $Sim(\ China, Korea)$ and $Sim(Korea, China)$. We found that $Sim(\ China, Korea)$< $Sim(Korea, China)$ in 40.8% of all times in 100,000 repetitions, with another 18.4% of all times corresponding to exact equalities (the simulation does not always produce equalities because of rounding errors). Likewise, when the subspaces for both China and Korea corresponded to random planes, we found that $Sim(China, Korea)$< $Sim(Korea, China)$ in 35.8% of all times in 100,000 repetitions, with 28.2% of all cases corresponding to exact equalities.

We then explored a five-dimensional space, with China corresponding to a four-dimensional subspace and Korea to a random plane. In this case, instead of comparing a projection to a plane with a projection to a ray, we compared a projection to a four dimensional subspace (China) with a projection to a plane (Korea). Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ etc. be a basis set for this five dimensional subspace. Then, $P_{China} = |x_1\rangle\langle x_1| + |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3| + |x_4\rangle\langle x_4|$. Korea corresponded to a random plane and was specified as above. In 100,000 runs of this scheme, $Sim(China, Korea)$< $Sim(Korea, China)$ in 100% of all times.

Overall, a violation of symmetry in similarity judgments, in the predicted direction, emerges naturally from the quantum model, just by assuming that the dimensionality of the China subspace is larger than that of the Korea subspace. In the quantum model this means that we know more about China than Korea. To produce a similarity asymmetry, no parameters were manipulated, nor did we require an assumption about the target or the referent being more salient.

Bowdle and Medin (2001; cf. Rosch, 1975) suggested that the statement ‘Korea is similar to China’ is preferred because it is more informative than the converse statement. Bowdle and Medin (2001) further assumed that naïve observers typically prefer informative statements (cf. Grice, 1975) and that in considering a statement like ‘A is similar to B’ “information flows directionally from base [B]to target [A]” (p. 121). So, ‘Korea is similar to China’ will be preferred to ‘China is similar to Korea’, because in the former case the target (Korea) is a more deviant item and the base (China) is more like a reference point, which means that there is more potential for new inferences regarding Korea, on the basis of knowledge for China. This is similar to how the quantum model works. In computing $Sim(Korea, China)$, we project from the Korea to the China subspace, hence we can understand/interpret the Korea properties, with the more extensive set of China properties (basis vectors). By contrast, when projecting from China to Korea (for $Sim(China, Korea)$), the state vector can be understood in a more limited way, with the more limited set of Korea properties.

-------------------FIGURE 2 ABOUT HERE--------------------

IV. Triangle inequality

Tversky (1977) considered how similarity judgments can lead to violations of the triangle inequality, another one of the metric axioms. The triangle inequality can be expressed as *Distance* (A, B) < *Distance* (A, C) + *Distance* (C, B). If we equate distance with dissimilarity, and assume similarity is the negative of dissimilarity, then the triangle inequality states that *Dissimilarity* (A, B) would always be less than *Dissimilarity* (A, C) plus *Dissimilarity* (C, B) or that *Similarity* (A, B) would always be greater than *Similarity* (A, C) plus *Similarity* (C, B). Tversky (1977) reported an example where the latter relation is violated. Consider A=Russia and B=Jamaica, so that *Similarity* (A, B) = *Similarity* (Russia, Jamaica) is low. Consider also C=Cuba. But, *Similarity* (A, C) = *Similarity* (Russia, Cuba) is high (because of political affiliation) and *Similarity* (C, B) = *Similarity* (Cuba, Jamaica) is also high (because of geographical proximity). Thus, Tversky's example shows that *Similarity* (Russia, Jamaica) < *Similarity* (Russia, Cuba) + *Similarity* (Cuba, Jamaica), which suggests a violation of the triangle inequality (Figure 3). Note that this demonstration does not depend on an assumption that one country is more salient than the others.

If one employs an exponentially decaying function to link distance and similarity (e.g., Nosofsky, 1984; Shepard, 1987), then similarities can violate the triangle inequality, even if the underlying distances obey the triangle inequality. For example, consider *Distance* (A,B)=5 units, *Distance* (A,C)=4 units, and *Distance* (C,B)=4 units; these distances obey the triangle inequality. For the similarities to follow Tversky's results we need that *Similarity*(A,B) < *Similarity*(A,C) + *Similarity*(C,B), and this relation is obtained from $e^{-5} = 0.0067 < e^{-4} + e^{-4} = 0.018 + 0.018$. Alternatively, shifts in attention across comparisons can also lead to violations of the triangle inequality, even if dissimilarity is equated with (just) distance (Nosofsky, 1984). So, unlike for the case of violations of symmetry, violations of the triangle inequality do not present a challenge for basic geometric approaches to similarity. However, the QP similarity model does not link distance and similarity via an exponentially decaying function, nor do we employ an attention mechanism, so the question remains whether it is consistent with Tversky's (1977) so-called violations of the triangle inequality.

The application of the quantum similarity model only requires to specify the various concepts (Russia, Jamaica, Cuba; also, Communist, in the Caribbean sea) in a way consistent with Tversky's (1977) assumptions. First, we make the simplifying assumption that all concepts are represented with one-dimensional subspaces. Second, the concepts Communist and Not communist have to be orthogonal to each other, since a country cannot be both Communist and Not communist. Third, the Caribbean and Not Caribbean concepts (corresponding to countries in the Caribbean or not) are specified in an analogous way. Fourth, the Communist concept is assumed to be unrelated to the Caribbean one, therefore, the one-dimensional subspaces corresponding to these two concepts are at an approximately 45^0 angle with respect to each other. This means that, if the state vector is, for example, in the Communist subspace, then the length of the projection to the Caribbean ray is the same as to the Not Caribbean ray. In other words, knowing that a country is communist is uninformative regarding whether the country is in the Caribbean.

Specifying the Communist and Caribbean concepts allows us to specify the Russia, Cuba, and Jamaica ones. Congruency between a country and a property is reflected in the proximity between the corresponding subspaces. Thus, the Russia subspace would have to be close to the Communist subspace (Tversky's experiment was done in the 70s) and away from the Caribbean subspace. Then, projecting from the Russia subspace to the Communist one would lead to a large projection, in the same way as for the quantum computation regarding the similarity between two concepts. The Cuba subspace is in-between the Communist and the Caribbean one and the Jamaica one is near the

Caribbean one, but far from the Communist one (Figure 4). Thus, different properties for the three countries are implied by proximity to the subspaces corresponding to these properties. Also, the two countries which are near the Communist subspace would both have large projections onto the Communist subspace and, therefore, they would be similar to each other by virtue of the fact that they both share the property of Communism.

Note that, as we are dealing with subspaces of the same dimensionality, any similarity computations here are insensitive to order (one order would not systematically produce a higher similarity than the opposite order, with an unbiased initial state, when averaging across a random sample of projector pairs). Note also that, as before, we require that the state vector (the activated thoughts just prior to the similarity comparisons) is set up so that $|P_{Cuba}|\psi\rangle|^2 = |P_{Russia}|\psi\rangle|^2 = |P_{Jamaica}|\psi\rangle|^2$, that is, the state vector is not biased towards any of the countries. But, in this case, rather than compute the initial state vector explicitly, which would require a dimensionality greater than two, we assume for simplicity that $Sim(Russia, Cuba) = |P_{Cuba}|\psi_{Russia}\rangle|^2$, whereby $|\psi_{Russia}\rangle = |Russia\rangle$, and likewise S$im(Cuba, Jamaica) = |P_{Jamaica}|\psi_{Cuba}\rangle|^2$, whereby $|\psi_{Cuba}\rangle = |Cuba\rangle$, and $Sim(Russia, Jamaica) = |P_{Jamaica}|\psi_{Russia}\rangle|^2$. This approach just affects the overall scaling of the similarity results. To account for the findings regarding the triangle inequality we require $|P_{Cuba}|\psi_{Russia}\rangle|^2$+$|P_{Jamaica}|\psi_{Cuba}\rangle|^2$>$|P_{Jamaica}|\psi_{Russia}\rangle|^2$.

Let us denote the angle between Russia and Cuba as ϑ_{RC} and likewise for the angle between Cuba and Jamaica (ϑ_{CJ}). Note also that, as all countries correspond to unidimensional subspaces, for example, $|P_{Cuba}|\psi_{Russia}\rangle|^2 = ||Cuba\rangle\langle Cuba|\psi_{Russia}\rangle|^2$, whereby $\langle Cuba|\psi_{Russia}\rangle = cos\vartheta_{RC}$ and $||Cuba\rangle|^2 = 1$. Starting from the condition required to account for Tversky's (1977) related finding, $|P_{Cuba}|\psi_{Russia}\rangle|^2$+$|P_{Jamaica}|\psi_{Cuba}\rangle|^2$>$|P_{Jamaica}|\psi_{Russia}\rangle|^2$, we must have $cos^2\vartheta_{RC} + cos^2\vartheta_{CJ} > cos^2\vartheta_{RJ}$ or $cos^2\vartheta_{RC} + cos^2\vartheta_{CJ} > cos^2(\vartheta_{RC} + \vartheta_{CJ})$. For angles up to 90^0, the cosine function is monotonically decreasing. Therefore, the condition $cos^2\vartheta_{RC} + cos^2\vartheta_{CJ} > cos^2(\vartheta_{RC} + \vartheta_{CJ})$ will be true, as long as ϑ_{RC} and ϑ_{CJ} are in the [0, 45^0] range, i.e., as long as Russia is similar to Cuba and Cuba is similar to Jamaica (it does not matter whether Russia is more or less similar to Cuba, than Cuba is to Jamaica). For illustration, we created a set of vectors corresponding to the ones in Figure 4. Russia was 5^0 counterclockwise from the Communist ray, Jamaica 5^0 clockwise from the Caribbean ray, and Cuba halfway between the Communist and the Caribbean rays (i.e., 67.5^0 relative to the horizontal). We obtained $|P_{Cuba}|\psi_{Russia}\rangle|^2 = cos^2\vartheta_{RC} = \cos{(27.5^0)}^2 = 0.79$, $|P_{Jamaica}|\psi_{Cuba}\rangle|^2 = 0.79$, and $|P_{Jamaica}|\psi_{Russia}\rangle|^2 = \cos^2{(\vartheta_{RC} + \vartheta_{CJ})} = \cos{(55^0)}^2 = 0.33$, thus reproducing Tversky's (1977) finding regarding the violation of the triangle inequality.

It is straightforward to see why the quantum model produces violations of the triangle inequality. Different regions in the knowledge space reflect different properties. Cuba is at the boundary between the Communist and Caribbean regions. Proximity to the Communism region makes it similar to Russia and to the Caribbean region to Jamaica. As the dimensionality of the knowledge space increases, more intricate patterns of similarity can emerge.

Tversky's (1977) own proposal was that different similarity comparisons are based on different properties. For example, the similarity between Russia and Cuba is based on the Communism property. Thus, the basic idea is not unlike the way the quantum model works, since proximity in different regions of knowledge space implies similarity on the basis of likewise different properties (e.g., proximity in the Communism part of the knowledge space implies similarity on the basis of Communism). However, there is an important difference between Tversky's (1977) and the quantum model. In Tversky's (1977) model, one needs to invoke particular features, which plausibly guide the similarity between two concepts. Thus, in comparing Russia and Cuba, the Communism feature is invoked. However, this mechanism is underspecified: why is only the Communism feature invoked? Why not also features relating to trade ties or similar political leaders? In fact, there is an infinite number of possible features one could invoke. We are just re-expressing Goodman's (1972) concerns regarding the arbitrariness of similarity, though we do not think the problem is with similarity, rather it has to do with guessing features. The quantum model avoids this problem: proximity in psychological space *implies* the existence of common features, but identifying these features is not required for the similarity computation.

-------------------FIGURES 4, 5 ABOUT HERE--------------------

V. Diagnosticity effect

This finding concerns the context dependence of similarity relations. Tversky (1977) employed a forced choice similarity task, whereby participants were asked to decide which country was most similar to Austria, amongst a set of candidate choices. Performance in such a task clearly depends on the pairwise similarity between the target country, Austria, and each of the candidate countries. Equally, each pairwise similarity may be affected by the alternative choices. Indeed, when the candidate choices were Sweden, Hungary, and Poland, participants tended to select Sweden as most similar to Austria (49% of participants favored this choice). When the candidate choices were Sweden, Norway, and Hungary, participants, tended to select Hungary as most similar to Austria (60% of participants favored this choice; analogous demonstrations were provided with schematic stimuli). The exact task Tversky (1977) employed involved presenting two groups of participants with 20 sets of four countries. Participants were required to choose, for each set of four countries, the country in the set most similar to a target country. Regarding the critical comparisons between matched quadruples of countries (e.g., Austria, Sweden, Hungary, Poland vs. Austria, Sweden, Norway, Hungary), the design was a between-participants one. We consider how, in the set of countries Hungary, Poland, Sweden, and Austria, Sweden ends up being most similar to Austria, since the situation involving Hungary, Sweden, Norway, and Austria is entirely analogous.

Tversky's (1977) explanation for the diagnosticity effect was partly the idea that a diagnostic feature of Eastern vs. Western Europe emerges, and it is this feature that makes Sweden more similar to Austria, than Hungary and Poland. Because this approach is popular in the literature, we note that it can be expressed in quantum terms. Suppose there is an Austria ray in a 3D space, which is equidistant to Hungary/Poland rays and a Sweden ray. A suitable 2D space corresponding to the Eastern vs. Western feature could be identified, such that, when expressing the Sweden, Austria, Hungary, Poland rays with the basis of that subspace, the Sweden ray becomes most similar to the Austria one. But, per our discussion for the triangle inequality, the idea of emergent features is underspecified. For example, what determines the particular diagnostic features that emerge, why would there be only one pair of (Eastern, Western Europe) diagnostic features, and how does the similarity between the countries in the comparison moderate the emergence of diagnostic features? Therefore this explanation is questionable, and the findings demand an alternative approach.

Tversky (1977) selected his materials so that two options are grouped together (Hungary, Poland) and these are approximately equally similar to the target (Austria) as the third option (Sweden). Representing these concepts in a quantum knowledge space and computing similarity in a way that takes into account the relevant context leads to a diagnosticity effect.

We seek the country which maximizes the similarity with the target, that is, the one for which $Sim(Chosen\ country, Target\ country) = max$. In this similarity comparison, the state vector $|\psi\rangle$ would be influenced by the alternative possibilities, since these are possibilities directly relevant to the similarity comparison. Therefore, when examining the similarity between Austria and a candidate country, the remaining two alternatives constitute a *context* of relevant information. For example, $Sim(Sweden, Austria) = |P_A P_S|\psi'\rangle|^2$, but where $|\psi'\rangle$ reflects some influence from the other possibilities of Hungary and Poland (note the subscripts A, S, H, and P, correspond to Austria, Sweden, Hungary, and Poland). So, $|\psi'\rangle = P_H P_P|\psi\rangle$ or $|\psi'\rangle = P_P P_H|\psi\rangle$. That is, the vector relevant in the comparison between Sweden and Austria is the one produced by first thinking of Hungary and Poland, so that $Sim(Sweden, Austria\ ) = |P_A P_S P_H P_P|\psi\rangle|^2$ or by first thinking of Poland and then Hungary, so that $Sim(Sweden, Austria\ ) = |P_A P_S P_P P_H|\psi\rangle|^2$. The context elements would influence the comparison in one order for some participants (e.g., $P_H P_P$) and in another order for others ($P_P P_H$). Averaging across the two possible orders for contextual influence, we have $Sim(Sweden, Austria\ ) = \frac{(|P_A P_S P_H P_P|\psi\rangle|^2 + |P_A P_S P_P P_H|\psi\rangle|^2)}{2}$.

The country most similar to the target Austria is identified by comparing $Sim(Sweden, Austria\ )$, with $Sim(Poland, Austria\ )$ and $Sim(Hungary, Austria\ )$. Specifically, we are led to
$Sim(Sweden, Austria) = \frac{(|P_A P_S P_H P_P|\psi\rangle|^2 + |P_A P_S P_P P_H|\psi\rangle|^2)}{2}$, and likewise
$Sim(Hungary, Austria) = \frac{(|P_A P_H P_S P_P|\psi\rangle|^2 + |P_A P_H P_P P_S|\psi\rangle|^2)}{2}$ and
$Sim(Poland, Austria\ ) = \frac{(|P_A P_P P_S P_H|\psi\rangle|^2 + |P_A P_P P_H P_S|\psi\rangle|^2)}{2}$. The empirical result is that $Sim(Sweden, Austria)$ is highest amongst $Sim(Hungary, Austria)$ and $Sim(Poland, Austria\ )$, given the appropriate context for each comparison. We seek to reproduce the diagnosticity effect, only on the basis that some of the available choices are grouped together (Tversky, 1977).

As before, it simplifies computations if we assume that the countries are represented by unidimensional subspaces (there is no indication that we have greater knowledge for one country, as opposed to the others). Analytically (Appendix 3), in comparing $Sim(Sweden, Austria)$, $Sim(\ Hungary, Austria)$, and $Sim(\ Poland, Austria)$, we are comparing

$$cos^2\vartheta_{SH} \cdot cos^2\vartheta_{HP} + cos^2\vartheta_{SP} \cdot cos^2\vartheta_{PH}\ ,$$
$$cos^2\vartheta_{HS} \cdot cos^2\vartheta_{SP} + cos^2\vartheta_{HP} \cdot cos^2\vartheta_{PS}\ ,$$
$$cos^2\vartheta_{PS} \cdot cos^2\vartheta_{SH} + cos^2\vartheta_{PH} \cdot cos^2\vartheta_{HS}.$$

But, the only ‘high’ cosine terms are those which involve the angle between Hungary and Poland, since this is the only ‘small’ angle (Hungary and Poland are the only countries assumed to be similar to each other). It is clear that $Sim(Sweden, Austria)$ is the only term which involves two high cosine terms, while each of the $Sim(Hungary, Austria)$ and $Sim(Poland, Austria)$ terms involve only one high cosine term. It follows that $Sim(Sweden, Austria)$ would be, on average, higher than $Sim(Hungary, Austria)$ and $Sim(Poland, Austria)$, as required for a demonstration of Tversky’s (1977) diagnosticity effect. Similar considerations can be made for when the task is to identify the country most similar to Hungary and the country most similar to Sweden (Appendix 3). For example, in the former case, we are comparing $Sim(Austria, Hungary)$, $Sim(Sweden, Hungary)$, and $Sim(Poland, Hungary)$. In this case, $Sim(Poland, Hungary)$ emerges as highest because the high cosine term (corresponding to the low angle between Poland and Hungary) appears only in the case of $Sim(Poland, Hungary)$.

It is straightforward to provide a computational illustration of the effect. Figure 5 shows a plausible geometrical arrangement for Austria, Sweden, Poland, and Hungary. As we did for the Russia-Cuba-Jamaica example, rather than directly compute an initial state vector which leads to the same projection for all four countries (that is, an initial state vector which is not biased towards any of the countries), it is simpler to assume that $P_{Sweden}|\psi\rangle = |Sweden\rangle$, $P_{Poland}|\psi\rangle = |Poland\rangle$ etc., that is, that the first projection to any of the countries is simply a normalized vector along the countries.

Let us define the Sweden angle to correspond to the angle between the Sweden ray and the horizontal, and likewise for the Austria angle, the Poland angle, and the Hungary angle. We set the Sweden angle equal to a random angle between 0 and 45^o, the Poland angle equal to the Sweden angle plus 90^o plus a random angle between -5^o and 5^o, the Hungary angle equal to the Poland angle plus a random angle between -5^o and 5^o, and the Austria angle equal to $\frac{(Hungary\ angle+Poland\ angle)/2+Sweden\ angle}{2}$ plus a random angle between -5^o and 5^o. Repeating this scheme 100,000 times, we found that the $Sim(Sweden, Austria)$ was highest in 77.1% of all times. This preference for Sweden is a demonstration consistent with Tversky's diagnosticity effect, since the context of Hungary and Poland make Austria and Sweden more similar to each other, than they would have been without this context. Without the context elements, $Sim(Sweden, Austria)$ was highest in approximately 44% of all cases, compared to $Sim(Hungary, Austria)$ and $Sim(Poland, Austria)$. Note that the model was set up so that Sweden is equally similar to Austria as Hungary/Poland are. Therefore, without context, about half the time Austria turned out to be most similar to Sweden and about half the time to either Poland or Hungary.

According to Tversky (1977), it is (partly) the implied classification of some of the alternatives, which produces the diagnosticity effect. We next explore the emergence of the diagnosticity effect, as the relation between the four countries is altered. Suppose that Hungary and Poland are dissimilar to each other and/ or Sweden is similar to Hungary and Poland. Under such circumstances, we expect the diagnosticity effect to break down. We verified that the QP model is consistent with this expectation. We examined model predictions across a range of values for the angle between the Sweden ray and the Poland ray (the Sweden, Poland offset angle) and the maximum allowed angle between the Hungary and Poland rays (the Hungary 'jiggle' angle). Figure 6 shows these results. It can be seen that the diagnosticity effect attenuates as the Hungary 'jiggle' angle increases and as the Sweden, Poland offset angle decreases. In other words, as the grouping of Hungary and Poland against all the other countries breaks down, so does the presence of the diagnosticity effect.

We illustrate the model in two additional ways. We examined its predictions when the task was to identify the country most similar to Hungary amongst Austria, Sweden, and Poland and the country most similar to Sweden amongst Austria, Poland, Hungary. Across 100,000 iterations, in 100% of all cases, the model predicted that Poland would be chosen as most similar to Hungary in the former case and Austria would be chosen as most similar to Sweden in the latter case. Both these results are consistent with expectation.

Tversky (1977, 343) insightfully observed that "The diagnosticity of features is determined by the classifications that are based on them." But, according to Tversky, the grouping of some alternatives, such as Hungary and Poland, leads to the emergence of particular diagnostic features, such as Eastern European country vs. not Eastern European country. Then, these diagnostic features make the remaining alternative more similar to the target country. But, as discussed, an explanation relying on diagnostic features can have problems.

In the quantum model, context corresponds to successive projections between the context elements. When the context elements are grouped together (as for Hungary, Poland), projecting across them leads to little loss of amplitude of the state vector, so that the similarity judgment ends up being higher. When there is no grouping across any of the possible contexts, then the effect of context is simply to uniformly scale the similarity judgments. So, context can make the same similarity comparison appear higher or lower, depending exactly on the grouping of the context elements. Finally, a grouping of alternatives means that there are subspaces along which these alternatives have a high projection, that is, properties or features that are common to some alternatives, but not others. So, overall, the intuition for how the quantum model produces the diagnosticity effect is not much different from that of Tversky's (1977). But, in Tversky's (1977) model it has to be assumed that diagnostic features are invoked, as a result of the grouping, while in the quantum model, the diagnosticity effect emerges directly from the presence of a grouping.

-------------------FIGURES 6,7 ABOUT HERE--------------------

VI. Conclusions and future directions

The objective of this paper was to generalize the notion of geometric representations. In the quantum proposal, the representation of an object need not be restricted to a single vector in a multidimensional psychological space, rather it can be a subspace of arbitrary dimensionality. The QP framework was developed to do exactly this, that is, associate knowledge with subspaces. The idea of representations as subspaces allows us to capture the intuition that a concept is the span of all the thoughts produced by combinations of the basic features that form the basis for the concept. Such relevant thoughts can include a central tendency, individual relevant instances, and properties. Moreover, the insight that concepts are about relevant thoughts, as well as instances, prototypes, etc., has been often expressed in discussions on representation and similarity (cf. Fodor, 1983; Murphy & Medin, 1985), but particular schemes for formalization have been lacking. Finally, the proposal for similarity is that this involves a process of thinking of the first and then the second of the compared entities. We were so able to cover some key empirical results: the basic violation of symmetry and the triangle inequality (Tversky, 1977) and the diagnosticity effect (Tversky, 1977).

One challenge for future work is to expand the range of empirical issues considered and motivate novel empirical demonstrations. For example, violations of symmetry can also arise from differences in perceptual salience or even frequency (Polk et al., 2002) and similarity judgments sometimes reflect correspondence between the parts of the compared stimuli (Larkey & Love, 2003; Markman & Gentner, 1993). More generally, a theoretical link between similarity and subjective probability has been the basis of influential ideas (Medin, Goldstone, & Markman, 1995; Shafir, Smith, & Osherson, 1990; Sloman, 1993; Tversky & Kahneman, 1983). Quantum theory provides a way to formalize this link. For example, in directional probability judgment tasks, one might seek to induce asymmetries in the consideration of predicates, analogous to the ones observed in similarity judgments.

The present emphasis was on the mathematical specification of the quantum model. One challenge for future work is detailed comparisons with alternative similarity models. We make some preliminary comments, in relation to symmetry. Tversky's (1977) own contrast model is that

$Similarity(A, B) = \theta f(A \cap B) - af(A - B) - \beta f(B - A)$, where θ, a, β are parameters, $A \cap B$ denotes the common features between A and B, A-B the features of A which B does not have and B-A the features of B which A does not have. Such a scheme can predict violations of symmetry if A has more features than B and the parameters a, β are different to each other and suitably set (e.g., $\theta > 0, a = 1, \beta = 0$). Tversky's model of similarity is fairly complex, as it involves two independent parameters. According to Tversky, these can be set by considering the relative salience of subject and referent in similarity judgments.

Krumhansl (1978) and Ashby and Perrin (1988) both provided sophisticated extensions to geometric models of similarity. In Krumhansl's (1978) proposal, the distance between two points *A* and *B* in psychological space is affected by the local density around each point, $D(A)$ and $D(B)$. The local density around a point reflects the number of other points within a certain radius. Thus, $d'(A, B) = d(A, B) + aD(A) + bD(B)$, where $d(A, B)$ is the standard geometric distance, *a* and *b* are weight parameters, and $d'(A, B)$ is the modified distance measure, as affected by local densities. In the case of the Korea-China example, for a violation of symmetry to occur, one needs to assume that the local density around China is different from the local density around Korea. Krumhansl (1978) suggested that prominent objects are likely to have many features and so these objects are likely to share features with a greater number of other objects, compared to objects with fewer features. Therefore, prominent objects are more likely to exist in denser regions of psychological space.

In Ashby and Perrin's (1988) general recognition theory, each stimulus (e.g., presented in different trials) can correspond to different points in psychological space, according to a particular probability distribution. Psychological space is divided into response regions, such that within each response region it is optimal to make a particular response. Then, similarity between two stimuli depends on the extent to which the distribution of perceptual effects for the first stimulus overlaps with the optimal response region for the second stimulus. For the Korea-China example, there are two factors predicting that $Sim(Korea, China) > Sim(China, Korea)$, first, that for many observers Korea will be a 'more vague and poorly defined concept' (p.133), so that the representation of Korea in psychological space will have a greater variability. Second, they argued that the response region for Korea would be smaller than that of China, because Korea is very similar to many other countries. But, observe that Krumhansl (1978, p.454) made the exact opposite assumption, that is, that it is China, not Korea, which is similar to a greater number of other countries. Thus, Ashby and Perrin (1988) and Krumhansl (1978) make opposite assumptions, regarding whether it is Korea or China, which is similar to a greater number of other countries. Of course, both Krumhansl's (1978) and Ashby and Perrin's (1988) models are sophisticated and a comprehensive examination requires more discussion. Our point is that coverage of even just symmetry violations in similarity is not straightforward.

Another fruitful comparison direction concerns models of information retrieval in large text-based corpuses. The quantum similarity model provides a way to assess the similarity between query and target, in a way that allows for asymmetries in the information content between the two and, also, can be informed by some proximal context (cf. Bruza, 2008). For example, an influential tool for information retrieval is Latent Semantic Analysis (LSA; Dumais, 2004). LSA works on a word by document matrix, with cells containing information about the frequency of words in the documents. Singular value decomposition is then employed to identify a set of dimensions, so that both the words and the documents can be represented in the same semantic space. Such a common space can be flexibly employed to compute similarities between words and documents or documents and other documents etc., usually in terms of the cosines of corresponding angles. A key difference between the LSA and the quantum similarity model is that in the former all documents, regardless of extent or complexity, would still be represented as single vectors. In the context of LSA applications, does it matter if more extensive documents are represented in a way equivalent to that of shorter ones? A related issue is that the similarity metric in LSA is symmetric, so the method would fare poorly with Tversky's (1977) key results (but see Griffiths, Steyvers, & Tenenbaum, 2007, for a generalization of these ideas, in a way allowing violations of the metric axioms). Again, one can ask whether empirical results in LSA application reveal any asymmetries or not. Moreover, LSA provides a data-driven mechanism for creating representations. If this work can be adapted to the specification of subspaces, instead of individual vectors, then this would enable a major development in the quantum similarity model.

It is natural to compare applications of quantum theory to psychology with ones of classical probability (CP) theory, since both are general, formal frameworks for assigning probabilities to events. Classical models have had an enormous influence in psychological theory (Chater, Tenenbaum, Yuille, 2006; Griffiths et al., 2010; Oaksford & Chater, 2007; Tenenbaum et al, 2011) and many researchers recognize the appeal of the kind of psychological explanation provided by CP models. Researchers interested in the application of QP theory in cognition aim to develop models with the same general characteristics as CP models. Specifically, quantum models aspire to cognitive explanations emphasizing the nature of representations, the operations on these representations, and the identification of the computational biases which guide cognitive process (Griffiths et al., 2010). Because of the sequential nature of projection in quantum theory, it will perhaps be easier to extend quantum models to include process assumptions, than it is generally the case for CP models (Jones & Love, 2011).

Quantum and classical probability theories arise from sharply different axiomatic foundations and so quantum theory has many unique characteristics, which have no analogue in classical theory. Notably, in quantum theory, computation can be order and context dependent and states are often superposition states, relative to the outcomes of a question. A superposition state vector cannot be said to possess a specific value for any of these possible outcomes and possible outcomes may interfere with each other, as the state vector develops in time. These features of quantum theory have enabled probabilistic models for situations which have been puzzling from a classical perspective (Aerts, 2009; Atmanspacher et al., 2004; Blutner, 2008; Bruza, 2010; Busemeyer & Bruza, 2012; Khrennikov, 2004; Yukalov & Sornette, 2010). A general difference between quantum and classical theories is that the latter require that there is always a complete joint probability distribution for all the questions relevant for a system (this is the principle of unicity; Griffiths, 2003). We suggest that such a requirement is psychologically unrealistic and, rather, the perspective dependent nature of calculation in quantum theory provides a more plausible framework for cognitive modeling (e.g., Pothos & Busemeyer, in press).

In this vein, regarding the quantum similarity model, its foremost characteristic is its sensitivity to the order and context of evaluating projections (and so similarities). There is a growing realization that analogous order effects are common in psychology, which recommends further study of QP cognitive models. Indeed, researchers interested in cognitive modeling have been increasingly employing computational machinery from this important class of models (Aerts & Gabora, 2005; Atmanspacher, Filk, & Romer, 2004; Bruza, Busemeyer, & Gabora, 2009; Bruza, Busemeyer, & Gabora, 2009; Busemeyer, Wang, & Townsend, 2006; Busemeyer et al., 2011; Franco, 2009; Khrennikov, 2004; Pothos & Busemeyer, 2009; Van Rijsbergen, 2004). The present work adds to this effort. As discussed, important challenges remain. We hope that the current analyses will motivate the application of the model in more specific problems and its further elaboration.

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Footnotes

Footnote 1. Infinite dimensional spaces are routinely employed in QP theory. Such spaces, when subject to certain completeness properties (e.g., that the infinite sum $|x_1|^2 + |x_2|^2 + \cdots$ strongly converges, where x_1, x_2 etc. correspond to the amplitudes of the state vector along different basis vectors), are the famous Hilbert spaces. From a psychological point of view, an infinite dimensional space means that it is possible, in principle, to have an infinite number of possible kinds of knowledge.

Footnote 2. As noted, these dimensions are best understood as vectors which correlate with sets of features.

Appendix 1: Computing the initial state vector (ψ), so that it does not bias the similarity comparison towards China or Korea, when assessing the similarity between the two.

This requires that ψ has the same projection to the China and Korea subspaces, or that $|P_{korea} \cdot \psi|^2 = |P_{china} \cdot \psi|^2$. We first work this out in the case where Korea is a one-dimensional subspace (a ray) and China is a two-dimensional subspace (a plane). Let's say that $|\psi\rangle = |korea\rangle + a|china1\rangle + a|china2\rangle$, where $|korea\rangle$, $|china1\rangle$, etc. are the respective basis vectors (we will shortly abbreviate these to $|k\rangle$, $|c_1\rangle$, etc.). We want to find *a* so that the projections to the Korea and China subspaces are the same. Note that ψ may need some additional normalization and that $\{|korea\rangle$ and $|china1, 2\rangle\}$ do not constitute an orthonormal basis.

$$P_{korea} \cdot |\psi\rangle = |k\rangle + a\langle k|c_1\rangle|k\rangle + a\langle k|c_2\rangle|k\rangle = (1 + a\langle k|c_1\rangle + a\langle k|c_2\rangle)|k\rangle$$

Likewise,

$$P_{china} \cdot |\psi\rangle = \langle c_1|k\rangle|c_1\rangle + \langle c_2|k\rangle|c_2\rangle + a|c_1\rangle + a|c_2\rangle = (\langle c_1|k\rangle + a)|c_1\rangle + (\langle c_2|k\rangle + a)|c_2\rangle$$

(Noting that we are in a real space so that $\langle a|b\rangle = \langle b|a\rangle$)

$$|P_{korea} \cdot |\psi\rangle|^2 = 1 + a^2\langle k|c_1\rangle^2 + a^2\langle k|c_2\rangle^2 + 2a\langle k|c_1\rangle + 2a\langle k|c_2\rangle + 2a^2\langle k|c_1\rangle\langle k|c_2\rangle$$
$$|P_{china} \cdot |\psi\rangle|^2 = \langle c_1|k\rangle^2 + a^2 + 2a\langle c_1|k\rangle + \langle c_2|k\rangle^2 + a^2 + 2a\langle c_2|k\rangle$$

Equating:

$$a^2(\langle k|c_1\rangle^2 + \langle k|c_2\rangle^2 + 2\langle k|c_1\rangle\langle k|c_2\rangle - 2) = \langle c_1|k\rangle^2 + \langle c_2|k\rangle^2 - 1$$

We next work out a more general formula:
Let's say

$$P_{korea} = |k_1\rangle\langle k_1| + |k_2\rangle\langle k_2| + \cdots |k_k\rangle\langle k_k|$$
$$P_{china} = |c_1\rangle\langle c_1| + |c_2\rangle\langle c_2| + \cdots |c_c\rangle\langle c_c|$$

$k < c$ (*k* and *c* are the dimensionalities of the Korea and China subspaces respectively).

So that the initial state vector is:

$$|\psi\rangle = \sum_{i=1}^{k} |k_i\rangle + a\sum_{j=1}^{c} |c_j\rangle$$

$$P_{korea} \cdot |\psi\rangle = \sum_{l=1}^{k} P_{korea_l}\left\{\sum_{i=1}^{k} |k_i\rangle + a\sum_{j=1}^{c} |c_j\rangle\right\} = \sum_{i=1}^{k} |k_i\rangle + a\sum_{i=1}^{k} P_{korea_i}\sum_{j=1}^{c} |c_j\rangle$$
$$= \sum_{i=1}^{k} |k_i\rangle + \sum_{i=1}^{k}\sum_{j=1}^{c} a\langle k_i|c_j\rangle\,|k_i\rangle$$

$$P_{china} \cdot |\psi\rangle = \sum_{l=1}^{c} P_{china_l} \left\{ \sum_{i=1}^{k} |k_i\rangle + a \sum_{j=1}^{c} |c_j\rangle \right\} = a \sum_{i=1}^{c} |c_i\rangle + \sum_{i=1}^{c} P_{china_i} \sum_{j=1}^{k} |k_j\rangle$$
$$= a \sum_{i=1}^{c} |c_i\rangle + \sum_{i=1}^{c} \sum_{j=1}^{k} \langle c_i | k_j \rangle \, |c_i\rangle$$

From these expressions we have that

$$\sum_{i=1}^{k} \left(1 + \sum_{j=1}^{c} a \langle k_i | c_j \rangle \right)^2 = \sum_{i=1}^{c} \left(a + \sum_{j=1}^{k} \langle c_i | k_j \rangle \right)^2$$

$$\sum_{i=1}^{k} \left(1 + a^2 \sum_{j=1}^{c} \langle k_i | c_j \rangle^2 + 2a \sum_{j=1}^{c} \langle k_i | c_j \rangle + 2a^2 \sum_{l=1, m=l+1}^{c} \langle k_i | c_l \rangle \langle k_i | c_m \rangle \right) =$$
$$\sum_{i=1}^{c} \left(a^2 + \sum_{j=1}^{k} \langle c_i | k_j \rangle^2 + 2a \sum_{j=1}^{k} \langle c_i | k_j \rangle + 2 \sum_{l=1, m=l+1}^{k} \langle c_i | k_l \rangle \langle c_i | k_m \rangle \right)$$

Or:

$$k + a^2 \sum_{i=1}^{k} \sum_{j=1}^{c} \langle k_i | c_j \rangle^2 + 2a^2 \sum_{i=1}^{k} \sum_{l=1, m=l+1}^{c} \langle k_i | c_l \rangle \langle k_i | c_m \rangle =$$
$$c \cdot a^2 + \sum_{i=1}^{c} \sum_{j=1}^{k} \langle c_i | k_j \rangle^2 + 2 \sum_{i=1}^{c} \sum_{l=1, m=l+1}^{k} \langle c_i | k_l \rangle \langle c_i | k_m \rangle$$

Or:

$$a^2 \left(\sum_{i=1}^{k} \sum_{j=1}^{c} \langle k_i | c_j \rangle^2 + 2 \sum_{i=1}^{k} \sum_{l=1, m=l+1}^{c} \langle k_i | c_l \rangle \langle k_i | c_m \rangle - c \right)$$
$$= \sum_{i=1}^{c} \sum_{j=1}^{k} \langle c_i | k_j \rangle^2 + 2 \sum_{i=1}^{c} \sum_{l=1, m=l+1}^{k} \langle c_i | k_l \rangle \langle c_i | k_m \rangle - k$$

From this last equation, *a* can be computed.

Consider next how the value of the *a* parameter changes, depending on the relative dimensionality of the Korea, China subspaces. For example, in the case when both subspaces are assumed to be two-dimensional (this was one of the control demonstrations), $|\psi\rangle = |korea_1\rangle + korea_2\rangle + a \cdot (|china_1\rangle + |china_2\rangle)$. Thus, *a* reflects the extent to which the Korea basis vectors are weighted more or less than the China ones, so as to achieve the required condition of $|P_{China} \cdot |\psi\rangle|^2 = |P_{Korea} \cdot |\psi\rangle|^2$. In that case, when both China and Korea were planes, *a*>1 in 50.4% of all times and *a*<1 in 49.6% of all times, as expected (note that *a* values were computed across 100,000 specifications of random planes for Korea and China). For when China was modeled with a four-dimensional subspace and Korea with a two-dimensional one, $|\psi\rangle = |korea_1\rangle + korea_2\rangle + a \cdot$

$(|china_1\rangle + |china_2\rangle + |china_3\rangle + |china_4\rangle)$, so we would expect China basis vectors to be weighted less than the Korea ones. Indeed, we found that *a*<1 in 96.2% of all cases.

Appendix 2: An analytic examination of the quantum model for the Korea-China example, in the simple case in which Korea is a one-dimensional subspace and China a two-dimensional one.

Let us assume that the Korea subspace is spanned by $|k\rangle$ and the China one by $|c1\rangle$, $|c2\rangle$. Let us consider first $|P_{China}\psi_{Korea}|^2$, remembering that ψ_{Korea} is a normalized vector in the Korea subspace and so it can only be $|k\rangle$, since Korea is one dimensional. Then,

$$|P_{China}\psi_{Korea}|^2 = |(|c1\rangle\langle c1| + |c2\rangle\langle c2|)|k\rangle|^2 = ||c1\rangle\langle c1|k\rangle + |c2\rangle\langle c2|k\rangle|^2 = \langle c1|k\rangle^2 + \langle c2|k\rangle^2$$

In the converse case, ψ_{China} can be any vector of the form $a|c1\rangle + b|c2\rangle$, whereby $a^2 + b^2 = 1$. Then,

$$|P_{Korea}\psi_{China}|^2 = ||k\rangle\langle k|(a|c1\rangle + b|c2\rangle)|^2 = ||k\rangle(a\langle k|c1\rangle + b\langle k|c2\rangle)|^2 = (a\langle k|c1\rangle)^2 + (b\langle k|c2\rangle)^2 + 2ab\langle k|c1\rangle\langle k|c2\rangle$$

So, comparing $|P_{China}\psi_{Korea}|^2$ and $|P_{Korea}\psi_{China}|^2$ involves comparing $\langle c1|k\rangle^2 + \langle c2|k\rangle^2$ and $(a\langle k|c1\rangle)^2 + (b\langle k|c2\rangle)^2 + 2ab\langle k|c1\rangle\langle k|c2\rangle$. To acquire some further insight into these computations we can make the simplifying assumption that $\langle c1|k\rangle^2 = \langle c2|k\rangle^2 = \langle k|c1\rangle^2 = \langle k|c2\rangle^2$. Then, comparing $|P_{China}\psi_{Korea}|^2$ and $|P_{Korea}\psi_{China}|^2$ reduces to the comparison between $2\langle c1|k\rangle^2$ and $\langle k|c1\rangle^2 + 2a\sqrt{1-a^2}\langle k|c1\rangle^2$, where we used the fact that $b = \sqrt{1-a^2}$. It can be readily seen that $2a\sqrt{1-a^2}$ is always less than 1, unless *a* is exactly equal to *b*, and so, in the vast majority of cases $|P_{China}\psi_{Korea}|^2 > |P_{Korea}\psi_{China}|^2$. If $a = b = \frac{1}{\sqrt{2}}$, then $2a\sqrt{1-a^2} = \frac{2}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = 1$, that is, in this case $|P_{China}\psi_{Korea}|^2 = |P_{Korea}\psi_{China}|^2$. Thus, it can never be the case that $|P_{China}\psi_{Korea}|^2 < |P_{Korea}\psi_{China}|^2$.

Appendix 3: Analytic computations for the series of projections relevant to the demonstration for Tversky's (1977) diagnosticity effect.

We first consider the problem of identifying the country most similar to Austria, from the set of Sweden, Poland, Hungary (this was Tversky's original experiment for the diagnosticity effect). Let us consider, for example, $|P_A P_P P_H P_S|\psi\rangle|^2 = |P_A \psi_{PHS}|^2 |P_P \psi_{HS}|^2 |P_H \psi_S|^2 |P_S|\psi\rangle|^2$.
(A, P, H, S correspond to Austria, Poland, Hungary, Sweden, respectively.)

$|\psi_S\rangle = P_S|\psi\rangle = |S\rangle$. The initial state vector is defined so that its projection has the same magnitude along the rays corresponding to each of the four counties; for simplicity we can assume that this projection is equal in magnitude to the original vector, so that $|P_S|\psi\rangle|^2 = 1$. Noting, however, that more than two dimensions would be required to explicitly compute an appropriate initial state vector.

$P_H|\psi_S\rangle = |H\rangle\langle H|S\rangle = |H\rangle cos\vartheta_{HS}$ so that $|P_H|\psi_S\rangle|^2 = cos^2\vartheta_{HS}$. Note that, in general, $\langle a|b\rangle = |a||b|cos\theta$ only in real spaces, but throughout this work we assume real spaces (and also that all vectors are normalized to 1, as is standard in QP).
$|\psi_{HS}\rangle = |H\rangle$

$|\psi_{PHS}\rangle = P_P|\psi_{HS}\rangle = |P\rangle\langle P|H\rangle = |P\rangle cos\vartheta_{PH}$ so that $|P_P|\psi_{HS}\rangle|^2 = cos^2\vartheta_{PH}$
$|\psi_{PHS}\rangle = |P\rangle$

$P_A|\psi_{PHS}\rangle = |A\rangle\langle A|P\rangle = |A\rangle cos\vartheta_{AP}$ so that $|P_A|\psi_{PHS}\rangle|^2 = cos^2\vartheta_{AP}$

Thus, $|P_A P_P P_H P_S|\psi\rangle|^2 = cos^2\vartheta_{AP} \cdot cos^2\vartheta_{PH} \cdot cos^2\vartheta_{HS}$
Note the correspondence between the angle indices and the indices for the series of projections. Thus, any related projection terms can be straightforwardly computed. For example, $|P_A P_S P_H P_P|\psi\rangle|^2 = cos^2\vartheta_{AS} \cdot cos^2\vartheta_{SH} \cdot cos^2\vartheta_{HP}$

Thus,
$Sim(Austria, Sweden) = \frac{(|P_A P_S P_H P_P|\psi\rangle|^2 + |P_A P_S P_P P_H|\psi\rangle|^2)}{2}$ or

$$Sim(Austria, Sweden) = \frac{cos^2\vartheta_{AS} \cdot (cos^2\vartheta_{SH} \cdot cos^2\vartheta_{HP} + cos^2\vartheta_{SP} \cdot cos^2\vartheta_{PH})}{2}$$

$Sim(Austria, Hungary) = \frac{(|P_A P_H P_S P_P|\psi\rangle|^2 + |P_A P_H P_P P_S|\psi\rangle|^2)}{2}$ or

$$Sim(Austria, Hungary) = \frac{cos^2\vartheta_{AH} \cdot (cos^2\vartheta_{HS} \cdot cos^2\vartheta_{SP} + cos^2\vartheta_{HP} \cdot cos^2\vartheta_{PS})}{2}$$

$Sim(Austria, Poland) = \frac{(|P_A P_P P_S P_H|\psi\rangle|^2 + |P_A P_P P_H P_S|\psi\rangle|^2)}{2}$ or

$$Sim(Austria, Poland) = \frac{cos^2\vartheta_{AP} \cdot (cos^2\vartheta_{PS} \cdot cos^2\vartheta_{SH} + cos^2\vartheta_{PH} \cdot cos^2\vartheta_{HS})}{2}$$

Note, finally, that the model assumes that the Austria ray is approximately halfway the Poland/Hungary rays and the Sweden one. That is, we also have $cos^2\vartheta_{AS} \approx cos^2\vartheta_{AH} \approx cos^2\vartheta_{AP}$.

We now apply the above computation in the two related problems of, first, identifying the country most similar to Hungary from the set of Austria, Sweden, Poland and, second, identifying the country most similar to Sweden from the set of Austria, Poland, Hungary.

Computing similarity to Hungary

$Sim(Austria, Hungary) = \frac{(|P_H P_A P_S P_P|\psi\rangle|^2 + |P_H P_A P_P P_S|\psi\rangle|^2)}{2}$ or

$$Sim(Austria, Hungary) = \frac{cos^2\vartheta_{HA} \cdot (cos^2\vartheta_{AS} \cdot cos^2\vartheta_{SP} + cos^2\vartheta_{AP} \cdot cos^2\vartheta_{PS})}{2}$$

$Sim(Sweden, Hungary) = \frac{(|P_H P_S P_P P_A|\psi\rangle|^2 + |P_H P_S P_A P_P|\psi\rangle|^2)}{2}$ or

$$Sim(Sweden, Hungary) = \frac{cos^2\vartheta_{HS} \cdot (cos^2\vartheta_{SP} \cdot cos^2\vartheta_{PA} + cos^2\vartheta_{SA} \cdot cos^2\vartheta_{AP})}{2}$$

$Sim(Poland, Hungary) = \frac{(|P_H P_P P_S P_A|\psi\rangle|^2 + |P_H P_P P_A P_S|\psi\rangle|^2)}{2}$ or

$$Sim(Poland, Hungary) = \frac{cos^2\vartheta_{HP} \cdot (cos^2\vartheta_{PS} \cdot cos^2\vartheta_{SA} + cos^2\vartheta_{PA} \cdot cos^2\vartheta_{AS})}{2}$$

The high cosine term (corresponding to the low angle between Poland and Hungary appears only in the case of Sim(Poland, Hungary), hence this is predicted to be the highest similarity.

Computing similarity to Sweden

$Sim(Austria, Sweden) = \frac{(|P_S P_A P_H P_P|\psi\rangle|^2 + |P_S P_A P_P P_H|\psi\rangle|^2)}{2}$ or

$$Sim(Austria, Sweden) = \frac{cos^2\vartheta_{SA} \cdot (cos^2\vartheta_{AH} \cdot cos^2\vartheta_{HP} + cos^2\vartheta_{AP} \cdot cos^2\vartheta_{PH})}{2}$$

$Sim(Hungary, Sweden) = \frac{(|P_S P_H P_P P_A|\psi\rangle|^2 + |P_S P_H P_A P_P|\psi\rangle|^2)}{2}$ or

$$Sim(Hungary, Sweden) = \frac{cos^2\vartheta_{SH} \cdot (cos^2\vartheta_{HP} \cdot cos^2\vartheta_{PA} + cos^2\vartheta_{HA} \cdot cos^2\vartheta_{AP})}{2}$$

$Sim(Poland, Sweden) = \frac{(|P_S P_P P_H P_A|\psi\rangle|^2 + |P_S P_P P_A P_H|\psi\rangle|^2)}{2}$ or

$$Sim(Poland, Sweden) = \frac{cos^2\vartheta_{SP} \cdot (cos^2\vartheta_{PH} \cdot cos^2\vartheta_{HA} + cos^2\vartheta_{PA} \cdot cos^2\vartheta_{AH})}{2}$$

The relatively high cosine term (corresponding to the relatively low angle between Austria and Sweden appears only in the case of Sim(Austria, Sweden), hence this is predicted to be the highest similarity.

Figure captions

Figure 1. An illustration of the idea of projection.

Figure 2. The figure illustrates how a successive projection from a ray and then to a plane will preserve more amplitude than a successive projection to a plane and then to a ray. The first projection is assumed to retain the same amount of amplitude, regardless of whether it is to a ray or to a plane.

Figure 3. Tversky's (1977) example of when human similarity judgments can violate the triangle inequality, assuming that dissimilarity is some linear function of distance in a psychological space. The diagram implies that Similarity(Russia, Jamaica)> Similarity (Russia, Cuba) + Similarity (Cuba, Jamaica), but in practice the opposite is true.

Figure 4. A geometric, two-dimensional representation of the information in Tversky's Russia-Cuba-Jamaica example. The triangle inequality requires that *Dissimilarity* (Russia, Cuba) + *Dissimilarity* (Cuba, Jamaica) > *Dissimilarity* (Russia, Jamaica). The black perforated lines illustrate how the corresponding projections allow for *Similarity* (Russia, Cuba) (green) + *Similarity* (Cuba, Jamaica) (yellow) > *Similarity* (Russia, Jamaica) (blue), thus violating the triangle inequality.

Figure 5. Illustrating a model for Tversky's (1977) diagnosticity effect. The top panel shows the series of projections in $|P_A P_S P_H P_P \psi|^2$, corresponding to the similarity between Sweden and Austria (the subscripts correspond to Austria, Sweden, Hungary, and Poland). The bottom panel illustrates the series of projections in $|P_A P_P P_S P_H \psi|^2$, corresponding to the similarity between Poland and Austria. It can be seen that the overall result of the projection depends on whether the projections to Hungary and Poland are successive or not.

Figure 6. The horizontal axis correspond to the Poland Sweden offset angle and the vertical axis to the Hungary jiggle angle (please see text for explanation). The frequencies to compute the probabilities for each data point in the graph were computed across 10,000 iterations. Blue colors correspond to lower values and red colors to higher values. Thus, red values are more consistent with the presence of a diagnosticity effect.

Figure 1.

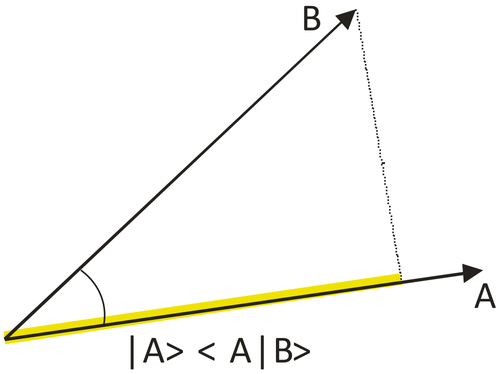
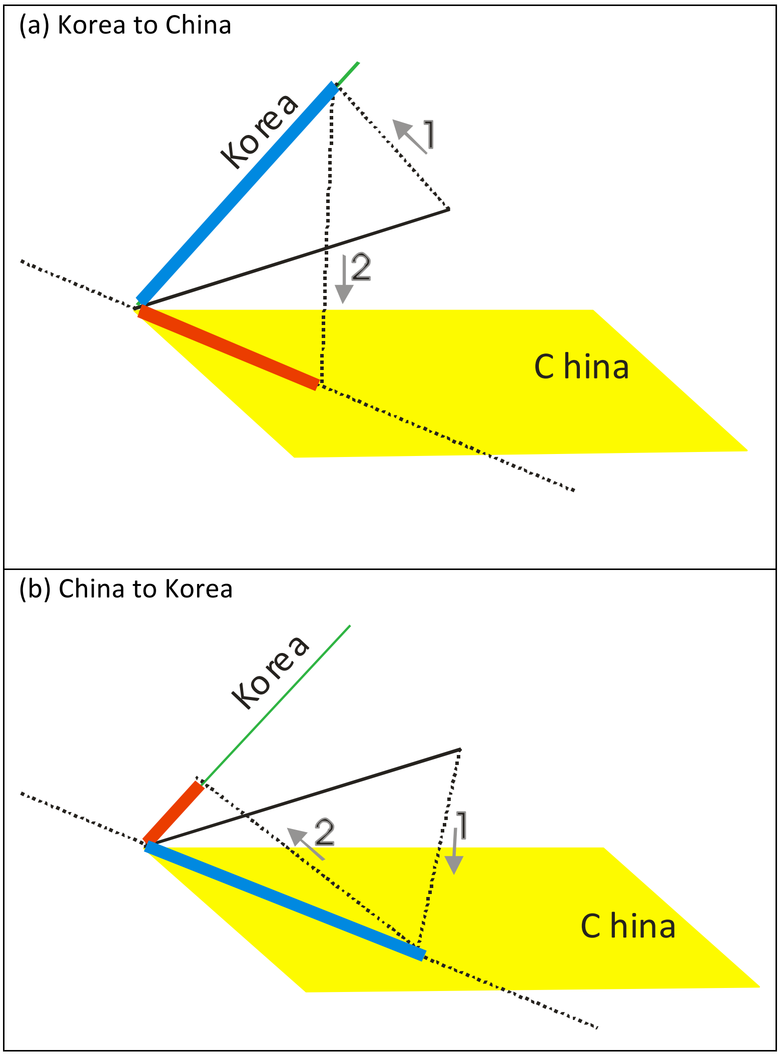


Figure 2.



(a) Korea to China

(b) China to Korea

Figure 3.

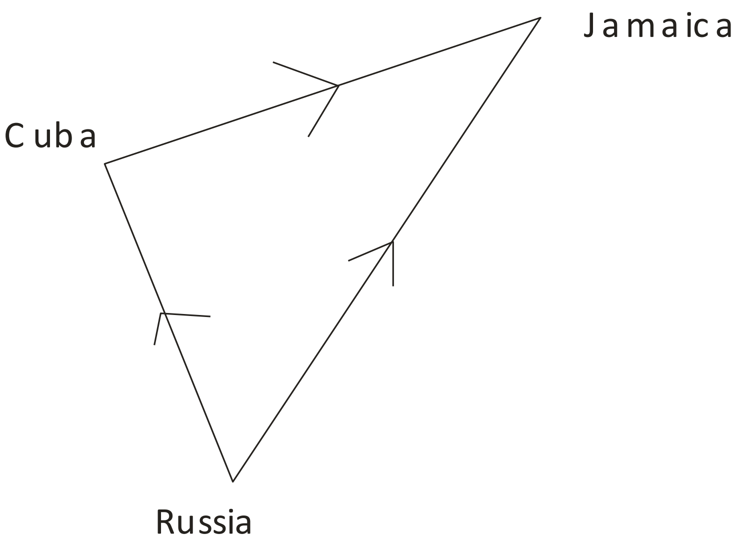


Figure 4.

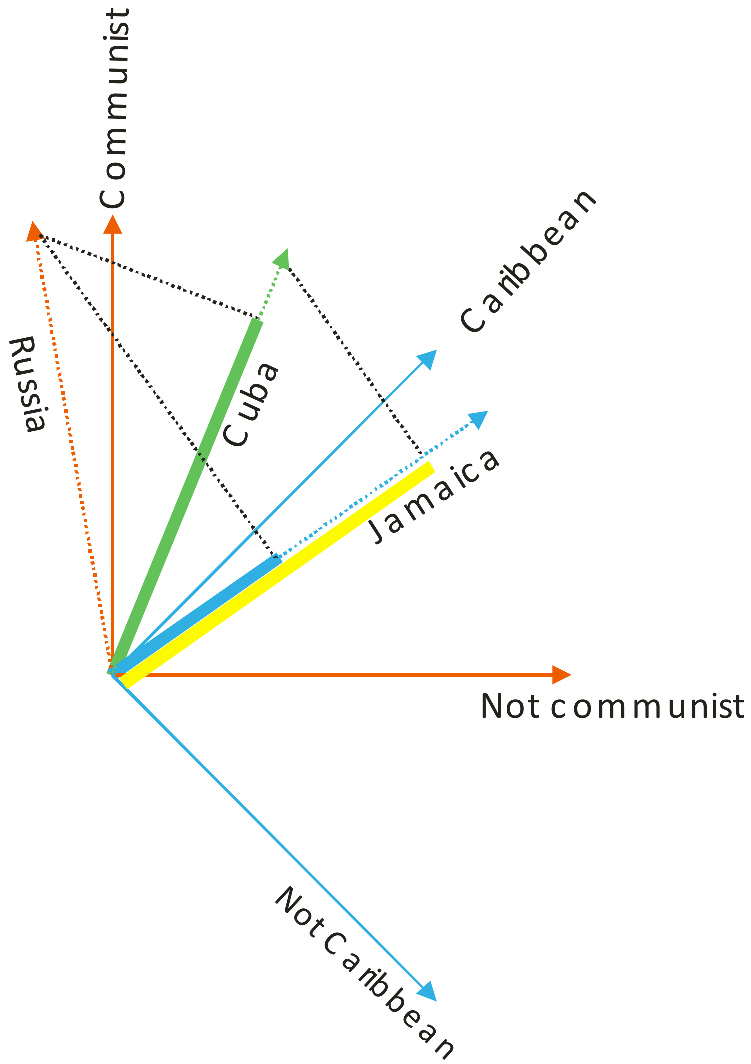

Communist
Russia
Cuba
Caribbean
Jamaica
Not communist
Not Caribbean

Figure 5.

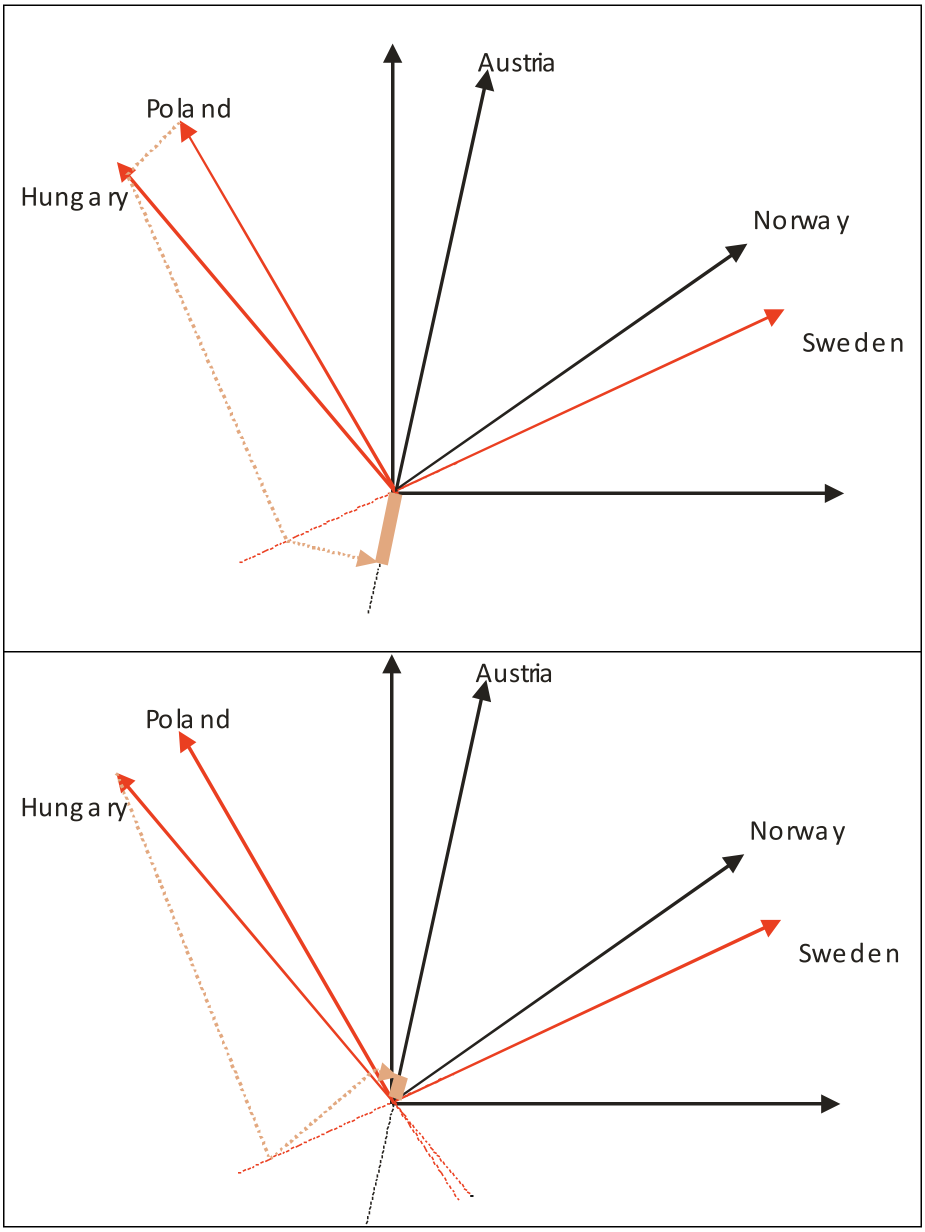
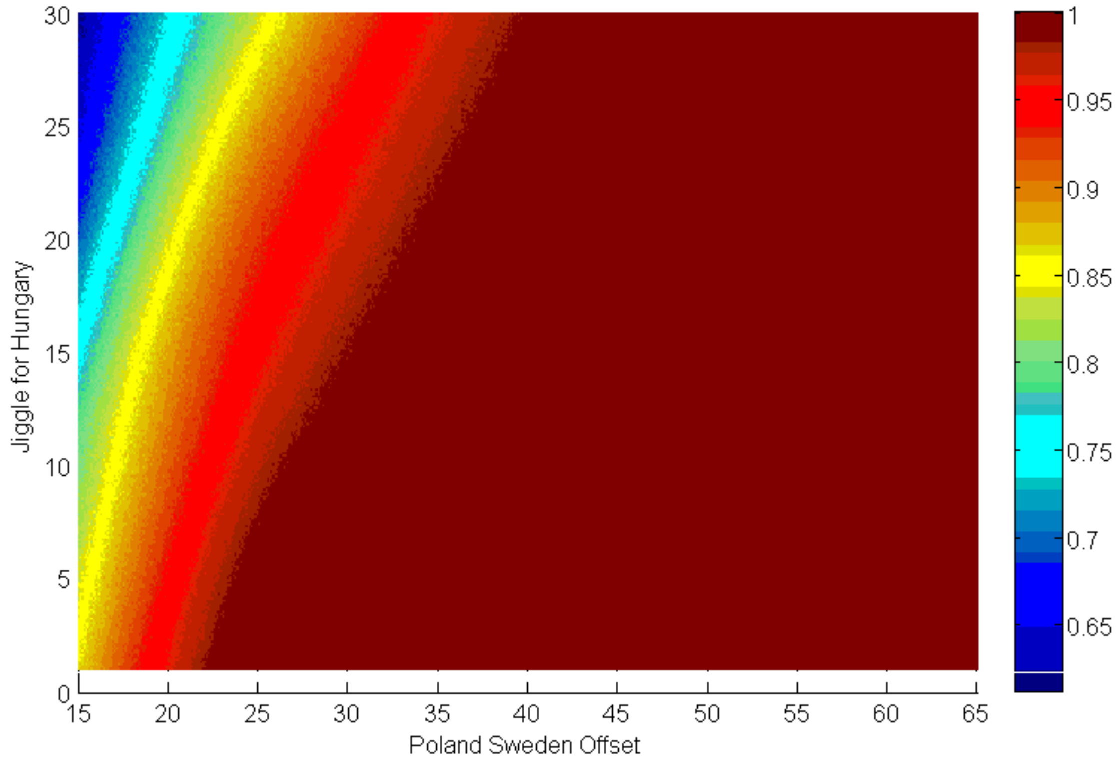

Austria
Poland
Hungary
Norway
Sweden
Austria
Poland
Hungary
Norway
Sweden

Figure 6.


Jiggle for Hungary
Poland Sweden Offset
0
5
10
15
20
25
30
15
20
25
30
35
40
45
50
55
60
65
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65