Section 4: Advanced Concepts and Methods

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Preamble

Will cover some advanced ideas needed to develop more realistic QT models.

Different philosophy from previous sections.

Want to give you a flavour of some of the issues, not an in depth look.

Idea is that if you come across these issues you’ll:

- Know something about them.
- Know where to look for more information.

Notes online [address]. Include more detail and references.

(Comments welcome!)
Introduction: Noise!

Material you’ve seen so far concerns idealised QT models.

i.e. perfect knowledge/control over states + measurement + evolution.

In the real world/lab life not this simple!

Need to adapt QT models to deal with noise.

Turns out this will also teach us some profound things about the meaning of QT.

So the theme of this section is ‘Noise’.
Plan for this Talk

1. Noise in the State: Density Matrices
2. Coffee Break!
3. Noise in the Measurement: POVMs
5. Summary
Noise in the State: Density Matrices
Suppose I’m doing an experiment and results depend on whether Ps left/right handed.

My PhD student collects an equal number of l and r handed Ps and lets them into lab one at a time.

Unfortunately he doesn’t tell me which ones are which! So 50/50 chance of getting a left or right handed P each time.

Suppose state of left handed Ps is $|L\rangle$ and that of right handed is $|R\rangle$, and they are orthogonal and complete.

What is correct state to describe my unknown Ps?
Might guess,

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|L\rangle + |R\rangle)$$  \hspace{1cm} (1)$$

Why? Well if I ask Ps if they are left handed,

$$p(left) = \langle \psi | P_L | \psi \rangle = \frac{1}{2} (\langle L | P_L | L \rangle + \langle L | P_L | R \rangle + \langle R | P_L | L \rangle + \langle R | P_L | R \rangle)$$

$$= \frac{1}{2}$$

But this isn’t right.

Eq.(1) is a **quantum** mixture, or **superposition**, but I wanted a **classical** mixture.

In other words, Eq.(1) says Ps **neither** r nor l handed, whereas of course each P is **definitely** one or the other, I just don’t know which.
The Density Matrix

Can we get a clue about the right answer by looking at statistics of measurement?

Suppose I’m measuring $O$, and expected result is $l,r$ for left/right handed Ps.

Since I have an equal number of $l$ and $r$ handed Ps, expected result is,

$$\langle O \rangle = \frac{1}{2} \langle L \mid O \mid L \rangle + \frac{1}{2} \langle R \mid O \mid R \rangle$$

$$= \frac{l + r}{2}$$

I can write this result in a simpler way by introducing the density matrix $\rho$,

$$\rho = \frac{1}{2} (\mid L \rangle \langle L \mid + \mid R \rangle \langle R \mid)$$
The Density Matrix

In terms of

$$\rho = \frac{1}{2} (|L\rangle \langle L| + |R\rangle \langle R|)$$

we can write the expectation value of $O$ as

$$\langle O \rangle = \text{Tr}(O \rho)$$

where $\text{Tr}$ denotes the trace of an operator, defined by,

$$\text{Tr}(A) = \sum_i \langle \phi_i | A | \phi_i \rangle$$

where $\{ \phi_i \}$ form an orthonormal basis for the Hilbert space.

Can show, 1) $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$, 2) $\text{Tr}(A)$ independent of basis. 3) $\text{Tr}(|\psi\rangle \langle \psi| A) = \langle \psi | A | \psi \rangle$. 
The Density Matrix

If we have a classical mixture of possible states $|\psi_\alpha\rangle$ which occur with probabilities $\omega_\alpha$ this ensemble can be represented by the density matrix,

$$\rho = \sum_\alpha \omega_\alpha |\psi_\alpha\rangle \langle \psi_\alpha|$$

Density matrices represent the most general way of writing QT. Every expression you’ve come across in QT has an analogue in terms of $\rho$.

Some properties of $\rho$,

- Hermitian, $\rho^\dagger = \rho$  \quad \rightarrow \quad \text{Eigenvalues real}
- Normalised, in the sense that $\text{Tr} (\rho) = 1$ \quad \rightarrow \quad \text{Eigenvalues sum to 1}
- Positive, meaning $\langle \psi | \rho | \psi \rangle \geq 0, \forall | \psi \rangle \in \mathcal{H}$ \quad \rightarrow \quad \text{Eigenvalues non-negative}
The Density Matrix

Useful to give some examples of QT in density matrix form.

1) Evolution, from $|\psi(t)\rangle = U(t) |\psi_0\rangle$ we get,

$$
\rho(t) = U(t) \rho_0 U^\dagger(t) \quad \leftrightarrow \quad \frac{\partial}{\partial t} \rho(t) = -i[H, \rho(t)]
$$

Known as a Master Equation.

2) Measurement. Probability we will get answer represented by $P_\alpha$ is given by,

$$
p(\alpha) = \text{Tr}(P_\alpha \rho)
$$

and if we do, state collapses to,

$$
\rho' = \frac{P_\alpha \rho P_\alpha}{\text{Tr}(P_\alpha \rho)}
$$

Convince yourself that setting $\rho = |\psi\rangle \langle \psi|$ recovers the results you know and love...
Let’s look again at our original question.

In the \{\ket{L}, \ket{R}\} basis our guess for the mixture state can be written as,

\[
\rho_g = \frac{1}{2}(\ket{L} + \ket{R})(\bra{L} + \bra{R}) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}
\]

while the correct answer is,

\[
\rho_c = \frac{1}{2}(\ket{L}\bra{L} + \ket{R}\bra{R}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}
\]

Two expressions differ only in ‘off-diagonal’ elements.

\(\rho_g\) is a quantum mixture (superposition), whereas \(\rho_c\) is a classical mixture.

So is the difference between classical and quantum states about the off-diagonal elements of \(\rho\)? Stay tuned...
How do I use a Density Matrix?

Most obvious use is modelling **individual differences**, in the sense of predicting distributions of behaviour, not just average.

1) If we **know** distribution of individual differences, can encode in $\rho$ and make predictions.

2) If we **don’t know** distribution, can leave as free parameter and use data to fit. Basically a QT mixed models approach.

Not been explored so far... but you are the future!
The Entropy of a Quantum State

Now we have states with classical and QT uncertainty, can ask about the Entropy. Entropy a very useful concept.

However it’s a bit complicated, so let’s start with an easier measure; Purity.

The Purity of a state is defined as,

\[ \gamma = \text{Tr}(\rho^2) \]

if \( \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \) for an orthonormal basis \( \{\phi_i\} \) then \( \gamma = \sum_i (p_i)^2 \)

Easy to see \( \gamma_g = 1 \), \( \gamma_c = \frac{1}{2} \).

States which can be written as \( \rho = |\psi\rangle \langle \psi| \) have \( \gamma = 1 \) and are called pure. States which cannot have \( \gamma < 1 \) and are called mixed.
The Entropy of a Quantum State

Now the classical Shannon entropy of a probability distribution \( \{p_i\} \) is,

\[
S_S = - \sum_i p_i \ln(p_i)
\]

If we write \( \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \) for orthonormal \( \{φ_i\} \), then in this basis we could define the analogue of the Shannon entropy as,

\[
S_{vN} = -\text{Tr}(\rho \ln(\rho)) \tag{2}
\]

But trace of an operator is basis independent, so Eq.(2) is valid in any basis. \( S_{vN} \) is known as the von Neumann entropy.

Pure states have \( S_{vN} = 0 \).
The Entropy of a Quantum State

Aside, $1 - \gamma$ is an approximate lower bound to $S_{vN}$, often called ‘Linear Entropy.’

Key advantage is that you don’t need to diagonalise $\rho$ to compute it, so much less computationally expensive.

For a classical probability distribution, max entropy state is uniform distribution.
Quantum analogue is density matrix,

$$\rho_{\text{Maximum Entropy}} = \frac{1}{d} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \frac{1}{d} I_d$$

where $I_d$ is identity matrix in $d$ dimensions. $S_{vN}(\rho_{\text{Maximum Entropy}}) = \ln(d)$. 
The Entropy of a Quantum State

As an example take state \( \rho = p |L\rangle \langle L| + (1 - p) |R\rangle \langle R|, \)
Haven’t discussed motivation/interpretation of QT much in this tutorial.

But this gives me an opportunity to mention one motivation for QT.

In classical probability, uncertainty about outcome means entropy in description. Entropy is interpretable as lack of information, or in other words,

\[ \text{Uncertainty about outcomes} \iff \text{Lack of knowledge about state} \]

Suppose I want to consider a model of my preference between two alternatives, \( A \) and \( B \), and suppose on average I’m indifferent between them.
Aside before Coffee

If I model my preference classically, means using classical probability distribution.

Classical probability → entropy → missing information. What does this imply?

Two perspectives:

Outside observer - Lacks information necessary to predict my choice. But what could this information be?

Ask “Do you have a definite preference for $A$ or $B.$” I say, “No.” Now what?

My perspective - I lack information about my preference? Can this make sense?

What if I’m certain about all relevant information, but still unsure about choice?
In contrast, in a quantum model one can have uncertainty about the outcomes, even though one has complete certainty about the state.

My choice could be uncertain and probabilistic, but there is no extra information you could acquire that would allow you to predict my choice any more accurately.

In other words, in QT,

Uncertainty about outcomes $\not\Rightarrow$ Lack of knowledge about state

Not forced to assume preferences have definite states, about which we’re ignorant.

For my money, this is one of the most powerful arguments for considering QT.
Noise in the State: Summary

- Introduced the density matrix; state with both classical and QT uncertainty.
- Given two groups of Ps, represented by the pure states $|P_1\rangle$ and $|P_2\rangle$, then we can form a mixed state by taking,
  \[ \rho = \lambda |P_1\rangle \langle P_1| + (1 - \lambda) |P_2\rangle \langle P_2|, \quad 0 \leq \lambda \leq 1. \]
- QT is most generally written in terms of density matrices.
- Discussed purity, $\gamma = \text{Tr}(\rho^2)$, and entropy, $S_{vN} = -\text{Tr}(\rho \ln(\rho))$. Pure states have zero entropy, $\rho$ lets us think about things like MaxEnt in QT.
- QT breaks the connection between uncertainty and entropy, and this might represent a powerful argument for use in cognition.
Coffee Break!

“It’s not as strong as methamphetamine, but it lets you keep your teeth.”

- Ross Scott
Noise in the Measurement: POVMs
Noise in the Measurement: Introduction

Most experiments contain some noise.

Could be intentional - time pressure in the task.

Could be unintended - Ps lose focus during long tasks, Ps not paying attention...

Want to model experiment where Ps express preference for one of \( A, B \).

**Ideal** measurement \( \implies \) represented by projection operators \( P_A = |A\rangle \langle A| \) etc.

What about a realistic noisy measurement?
Noise in the Measurement: Introduction

Noisy measurement means outcomes not perfectly aligned with preferences.

Suppose we represent measurement by operators $E_A$ and $E_B$, we want,

$$\langle A | E_A | A \rangle = 1 - \epsilon, \quad \langle B | E_A | B \rangle = \epsilon,$$

$$\langle A | E_B | A \rangle = \epsilon, \quad \langle B | E_B | B \rangle = 1 - \epsilon.$$

Where $0 \leq \epsilon \leq 1$ is some small ‘error’ probability.

In basis $\{|A\rangle, |B\rangle\}$ can therefore write $E_A, E_B$ as,

$$E_A = \begin{pmatrix} 1 - \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \quad E_B = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 - \epsilon \end{pmatrix}.$$
Noise in the Measurement: Introduction

\[ E_A = \begin{pmatrix} 1 - \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \quad E_B = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 - \epsilon \end{pmatrix}. \]

Can we use these to describe a \textit{measurement}?

They are \textbf{not} projection operators...

However, they are \textbf{positive} (positive eigenvalues) and \textbf{complete} \((E_A + E_B = 1)\).

Which means that for any \(\rho\),

\[ 0 \leq \text{Tr}(\rho E_i) \leq 1, \quad \text{for } i = \{A, B\} \quad \text{and} \quad \sum_{i=A,B} \text{Tr}(E_i \rho) = 1. \]

The quantities \(\text{Tr}(E_i \rho)\) can thus be interpreted as \textbf{probabilities}, so \(E_A, E_B\) are good candidates to describe a measurement.
Noise in the Measurement: Introduction

What sort of measurement do they describe?

One way to think about it is to note,

\[ E_A = (1 - \epsilon)P_A + \epsilon P_B, \quad E_B = \epsilon P_A + (1 - \epsilon)P_B. \]

In other words I can write these operators like,

\[ E_A = \sum_i p_A(i)P_i \]

where the \( p_A(i) \) have (loosely) the interpretation of probabilities.

Instead of performing \( P_A \) I randomly perform \( P_A \) or \( P_B \) with prob \( 1 - \epsilon \) and \( \epsilon \).

This is one sense in which \( E_A, E_B \) are noisy versions of \( P_A, P_B \).
$E_A, E_B$ are specific examples of positive operator valued measures or POVMs.

POVMs most general type of measurements in QT. Briefly outline general theory.

Most general description of a measurement process in QT is given by set of POVMs $\{E_i\}$, satisfying,

- Positivity, $\langle \psi | E_i | \psi \rangle \geq 0, \ \forall |\psi\rangle \in H$.
- Completeness, $\sum_i E_i = 1$.

Probability that a measurement described by $E_i$ gives a positive answer is,

$$p(i) = \text{Tr}(E_i \rho).$$
A family of POVMs can have many different realisations. Realisation $\phi_i$ is operation applied to state $\rho \rightarrow \phi_i(\rho)$ such that,

$$\text{Tr}(\phi_i(\rho)) = \text{Tr}(E_i\rho).$$

Simplest is to take the square root of $E_i$, so if $E_i = M_i^\dagger M_i$, then we write,

$$\phi_i(\rho) = M_i\rho M_i^\dagger.$$

$M_i$ often called ‘measurement operators’.

Collapse postulate: if measurement $E_i$ yields positive answer, then state collapses,

$$\rho' = \frac{\phi_i(\rho)}{\text{Tr}(\phi_i(\rho))} = \frac{M_i\rho M_i^\dagger}{\text{Tr}(E_i\rho)}.$$
POVMs

As mentioned, realisation is not unique! Ignore this here, but see notes for more.

In our example, for the POVM $E_A$, in the basis $\{ |A\rangle, |B\rangle \}$ the associated measurement operators will be,

$$M_A = \begin{pmatrix} \sqrt{1-\epsilon} & 0 \\ 0 & \sqrt{\epsilon} \end{pmatrix}, \quad M_B = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \sqrt{1-\epsilon} \end{pmatrix}$$

which are nice and simple.
Order Effects in Noisy Measurements

Order effects in surveys hugely important.

Analysis done with perfect measurements, do noisy measurements spoil this?

Simple example, initial state $|A\rangle$ and two possible projective measurements, $P_B$ onto $|B\rangle$ and $P_+$ onto $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$.

It is easy to see that

$$p(\text{+ and then } B) = \text{Tr}(P_B P_\rho P_+) = \frac{1}{4}$$

Whereas,

$$p(B \text{ and then +}) = \text{Tr}(P_+ P_B \rho P_B) = 0.$$ 

What happens if we replace the ideal measurements with POVMs?
Order Effects in Noisy Measurements

We let’s replace the projection operators with the following POVMs

\[ P_B \rightarrow E_B = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 - \epsilon \end{pmatrix}, \quad P_+ \rightarrow E_+ = \begin{pmatrix} \frac{1}{2} & \frac{1-2\epsilon}{2} \\ \frac{1-2\epsilon}{2} & \frac{1}{2} \end{pmatrix}. \]

These have associated measurement operators,

\[ M_B = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \sqrt{1-\epsilon} \end{pmatrix}, \quad M_+ = \begin{pmatrix} \frac{\sqrt{1-\epsilon}+\sqrt{\epsilon}}{2} & \frac{\sqrt{1-\epsilon}-\sqrt{\epsilon}}{2} \\ \frac{\sqrt{1-\epsilon}-\sqrt{\epsilon}}{2} & \frac{\sqrt{1-\epsilon}+\sqrt{\epsilon}}{2} \end{pmatrix}. \]

Now we can see that,

\[ p_\epsilon(+ \text{ and then } B) = \frac{1}{4} \left( 1 - 2(1 - 2\epsilon)\sqrt{\epsilon}\sqrt{1-\epsilon} \right) \]

\[ p_\epsilon(B \text{ and then } +) = \frac{\epsilon}{2}. \]
Order Effects in Noisy Measurements

Figure: Plotting $p_\epsilon(B \text{ and then } +)$, $p_\epsilon(+ \text{ and then } B)$ and their difference, against $\epsilon$.

Realistic measurements might have $\epsilon \sim 1\text{-}5\%$. 

One interesting property of POVMs, don’t have to be orthogonal.

Simple example, two choices $A$, $B$, but three possible responses, “prefer $A$”, “prefer $B$”, “don’t know”.

Represent with states,

$$\text{Prefer } A = |A\rangle, \quad \text{Prefer } B = |B\rangle$$

$$\text{Don’t know } = \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \text{ or } \frac{1}{\sqrt{2}}(|A\rangle - |B\rangle)$$

Have associated projection operators, $P_A, P_B, P_+, P_-$, but

$$\sum_{i=A,B,+,-} P_i = 2.$$ 

So can’t form a description of a measurement from $\{P_i\}$.
Application of POVMs: Non-Orthogonal Measurements

However can easily turn into POVMs by normalising,

\[ E_A = \frac{P_A}{2}, \quad E_B = \frac{P_B}{2}, \quad E_+ = \frac{P_+}{2}, \quad E_- = \frac{P_-}{2} \]

Suppose my state is \( |+\rangle = \frac{1}{\sqrt{2}} (|A\rangle + |B\rangle) \).

Then, \( p(A) = \frac{1}{4}, \quad p(B) = \frac{1}{4}, \quad p(\text{don't know}) = \frac{1}{2} \).

Not very realistic example (e.g. \( p(\text{don't know}) = \frac{1}{2} \) always!) but shows potential.

Interestingly, measurement of “don’t know” is non-selective...
Description of measurements in QT can be generalised to non-ideal measurements.

Can describe realistic noisy measurements, where even participants with a definite knowledge state may not make completely predictable decisions.

Can also model non-orthogonal sets of measurements.

POVMs likely to be a very useful tool as we strive to make QT predictions more accurate.
Noise in the Evolution: CP-Maps
Final type of noise we will consider is noise in the evolution of the state.

This sort of noise can arise for two reasons:

- Cognitive variables we are studying may interact with ones we are not.
- Effect of stimuli on different Ps may vary uncontrollably.

Turns out the correct way to model these two things is essentially the same.

Also turns out the be the same as the answer to the following question:

- How do we model evolutions that are irreversible, or have fixed points?

Unitary evolutions are reversible, but stimulus presentation often isn’t!
Noise in the Evolution: CP Maps

We want to study noisy evolutions in QT.

Generally speaking, these are motivated by considering a system of interest $S$ interacting with some less well controlled system which we call an environment $\mathcal{E}$.

We’ll go with this idea here, although you might have some concerns about the physics/cognition analogy.

In practice, one selects a given noisy evolution to do a particular job, without worrying about where it comes from. Main concern is that it has a standard form.

However I think it’s useful to have some idea about where they come from.
Noise in the Evolution: CP Maps

Full disclosure, this is wooly! Some extra math in the notes.

Idea: Want to study $S$, have little or no info about $E$, but know they interact.

We know full dynamics of $\rho_{S+E}$,

$$\frac{\partial}{\partial t} \rho_{S+E} = -i[H_{S+E}, \rho_{S+E}]$$

where $H_{S+E}$ is the joint Hamiltonian of the system plus environment.

Throw away info about $E$, leaving description of $\rho_S$.

What we end up with is,

$$\frac{\partial}{\partial t} \rho_S = -i[H_S, \rho_S] + \mathcal{L}\rho_S$$

where $\mathcal{L}$ is a super-operator encoding extra dynamics coming from $S$-$E$ interaction.
The most general form this equation can take is the so-called ‘Lindblad’ form,

\[ \frac{\partial}{\partial t} \rho_S = -i[H_S, \rho_S] + \sum_k \left( L_k \rho_S L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_S - \frac{1}{2} \rho_S L_k^\dagger L_k \right), \quad (4) \]

where \( \{L_k\} \) are ‘Lindblad’ operators, which model the effect of the environment.

**Promise** you will never need to use this in full generality!

Key feature of evolution according to Eq.(4) is that it preserves properties of \( \rho \), in particular positivity. Thus these evolutions called *Completely Positive* or *CP-maps*.

**Take home point:** If you can write your evolution in the form Eq.(4), then it’s a valid QT evolution.
A CP-Map for Irreversible Evolutions

Want to give an interesting example.

Suppose we have a simple 2D cognitive system spanned by \{\ket{1}, \ket{2}\}.

Initial state \(\psi_0 = \frac{1}{\sqrt{2}} (\ket{1} + \ket{2})\). Want to evolve the state in an irreversible way, giving Ps new information that cannot be taken back, so state tends towards \(\ket{1}\).

Unitary evolution won’t work:

- It’s reversible.
- Evolving for long enough will cause state to ‘overshoot’ back towards \(\ket{2}\).
One possible such evolution is given by,

$$\frac{\partial}{\partial t} \rho = \gamma \left( L \rho L^\dagger - \frac{1}{2} L^\dagger L \rho - \frac{1}{2} \rho L^\dagger L \right),$$

with $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and we've assumed there is no unitary part to the evolution.

The solution, for the initial condition we gave, is,

$$\rho(t) = \begin{pmatrix} 1 - \frac{1}{2} e^{-\gamma t} & \frac{1}{2} e^{-\frac{\gamma t}{2}} \\ \frac{1}{2} e^{-\frac{\gamma t}{2}} & \frac{1}{2} e^{-\gamma t} \end{pmatrix}$$

Tends to $|1\rangle$ at large times, and doesn't overshoot. $|1\rangle$ is therefore a fixed point.

Decays towards other states obtained by applying unitary transformation to $L$. 
A CP-Map for Irreversible Evolutions

One interesting property is that this evolution doesn’t preserve purity or entropy.

In summary, does a good job of modelling a more realistic, irreversible evolution.
A CP-Map for Uncertain Stimuli

Want to consider evolution caused by stimuli where effect on a given $P$ uncertain.

Suppose we present stimuli in continuously, so change in the state depends on the length of time for which the stimuli is presented.

Or stimuli are discrete, but individually weak, so that presentation of multiple stimuli approximated by a continuous in time master equation.

Either way, stimuli fixed, but effect on the participants unknown. Can assume stimuli on average produce a shift in a certain direction, but size of shift unknown.

Equivalent to definite evolution but unsure, for each $P$, how long state evolved for.
A CP-Map for Uncertain Stimuli

Specifically, assume effect of evolution of a state for time $t$ is not to cause,

$$\rho(0) \rightarrow e^{-iHt} \rho(0) e^{iHt}$$

but rather,

$$\rho(t) = \int ds \ p_t(s) \ e^{-iH(t+s)} \rho(0) e^{iH(t+s)}$$

$p_t(s)$ is a probability distribution of ‘evolution times,’ centred around 0.

If $p_t(s) = \delta(s)$ we recover standard unitary evolution.

$p_t(s)$ depends on $t$ because uncertainty, i.e. the width of $p_t(s)$, might depend on the time evolved for → longer average evolution times give larger uncertainties.
A CP-Map for Uncertain Stimuli

For boring reasons (see notes) form of $p_t(s)$ is highly constrained. One choice is,

$$p_t(s) = \sqrt{\frac{1}{\pi \sigma t}} e^{-\frac{s^2}{\sigma t}}$$

So we have invented a new type of evolution, given by,

$$\rho(t) = \int ds \, p_t(s) e^{-iH(t+s)} \rho(0) e^{iH(t+s)}, \quad \text{with } p_t(s) = \sqrt{\frac{1}{\pi \sigma t}} e^{-\frac{s^2}{\sigma t}}.$$

Is this a valid QT evolution?

Related question, what’s the differential form of this, so I can simulate it?
A CP-Map for Uncertain Stimuli

Turns out, using the very useful approximation,

\[ p_t(s) = \sqrt{\frac{1}{\pi \sigma t}} e^{-\frac{s^2}{\sigma t}} \approx \delta(s) + \frac{\sigma t}{4} \delta''(s) + \ldots \]

We can derive,

\[ \frac{\partial \rho}{\partial t} = -i[H, \rho] - \frac{\sigma}{4} [H, [H, \rho]] \]

\[ = -i[H, \rho] + \frac{\sigma}{2} (H \rho H - \frac{1}{2} H^2 \rho - \frac{1}{2} \rho H^2) \]

This is of Lindblad form, with \( L = \sqrt{\frac{\sigma}{2}} H \).

\[ \left( \text{Recall } \frac{\partial}{\partial t} \rho = -i[H, \rho] + \sum_k \left( L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k \right) \right) \]

Thus it’s a valid QT evolution!
A CP-Map for Uncertain Stimuli

What does it do?

Choose $H = \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and initial state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$.

The solution is,

$$\rho(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} e^{-2i\gamma t - \sigma \gamma^2 t} \\ \frac{1}{2} e^{2i\gamma t - \sigma \gamma^2 t} & \frac{1}{2} \end{pmatrix}$$

Recall our previous discussion. Identifying $|1\rangle \rightarrow |A\rangle$ etc.

$$\rho(0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \rho_g, \quad \rho(t \rightarrow \infty) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_c.$$

This evolution turns pure states into mixed states!
A CP-Map for Uncertain Stimuli

Look at probability of being in states $|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$.

Figure: Probability of being in state $|+\rangle$ (solid line) or state $|\rightarrow\rangle$ (dashed line).
We saw that the **noisy evolution** we introduced turns **pure** states into **mixed** ones.

In some sense this means turning **quantum superpositions** into **classical mixtures**.

This is actually a **general** result, which goes by the name of ‘Decoherence’.

Way to think of this is that quantum interference effects require a certain amount of **coherence** between alternatives, because **interference** is due to relative phase.

Adding noise **breaks** coherence, and washes out the quantum interference effects.

Important part of how things made of quantum particles can behave classically.

**Implications** for QT models?
Noise in the Evolution: Summary

Explored the idea of noisy evolutions in quantum cognition.

Idea: interactions between the system under study and other cognitive variables can influence the evolution of the system we are interested in in a profound way.

Most general type of evolutions in quantum cognition are called CP-maps can be written in the form of a master equation of Lindblad form.

Can use CP-maps to model more realistic irreversible evolutions.

Noisy evolutions have the effect of diminishing the ‘quantum-ness’ of a system.

- Might make it hard to see quantum effects in complex cognitive variables.
- Might also explain why some variables do not show any quantum effects.
Summary
In the real world/real lab, all cognitive processes involve some level of noise.

We’ve discussed how to incorporate noise into quantum systems via,
- The state (Density Matrix),
- The measurements (POVMs),
- The evolution (CP-Maps).

Also learned some practical lessons, such as how to compute the entropy of a state, how to model non-orthogonal measurements, and how to model evolution where we are unsure about the strength of a stimuli.
Summary: Realistic QT Models

- QT can be adapted to cover realistic experiments.
  - Inhomogeneous groups of participants.
  - Experimental error and noise
  - Interactions between the variables we would like to study, and those we are less interested in.
- Way open to better modelling, particularly of the statistical distribution of results for a given trial, rather than just average behaviour.
- Having an idea about how QT behaviour differs in presence of noise means we can also think about letting QT loose in the wild!
Summary: Robustness of QT Features

- Most QT features **robust** against **small** amounts of noise. (Reassuring).
- Equally, most QT features **break down** with **large** amounts of noise.
- Important implications for where we might see QT effects.
- Also important for understanding why **not all** variables show evidence for QT.
Summary: Adding Noise Teaches us Interesting Things

- Adding noise to QT teaches us more than just how to model careless Ps.
- Real structure of QT only apparent when we see POVMs and CP-maps.
- Classical and QT theories are closer than you might have thought.
- Adding noise lets you transition between them.
- Can we build a unified model of classical and QT decision making?
The end!

Thanks for coming to this tutorial and for listening!

Happy to take questions.

Remember, any comments on, or questions about, the notes, please send to james.yearsley.1@city.ac.uk