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COMMENT

The Conjunction Fallacy, Confirmation, and Quantum Theory: Comment on Tentori, Crupi, and Russo (2013)

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The conjunction fallacy refers to situations when a person judges a conjunction to be more likely than one of the individual conjuncts, which is a violation of a key property of classical probability theory. Recently, quantum probability (QP) theory has been proposed as a coherent account of these and many other findings on probability judgment "errors" that violate classical probability rules, including the conjunction fallacy. Tentori, Crupi, and Russo (2013) presented an alternative account of the conjunction fallacy based on the concept of inductive confirmation. They presented new empirical findings consistent with their account, and they also claimed that these results were inconsistent with the QP theory account. This comment proved that our QP model for the conjunction fallacy is completely consistent with the main empirical results from Tentori et al. (2013). Furthermore, we discuss experimental tests that can distinguish the 2 alternative accounts.

Keywords: decision making, conjunction fallacy, confirmation, quantum theory

This comment concerns a recent debate over formal explanations for the conjunction fallacy (Tversky & Kahneman, 1983). This fallacy occurs when a person judges the likelihood of the conjunctive event (A and B) to be greater than the likelihood of one of the events, say A, alone. The most well-known example is about a hypothetical person, Linda (L), who is described in such a way that she looks very much like a feminist (F) and not at all like a bank teller (B). Participants are asked to rank the relative likelihood of various statements about Linda, including the statement that "Linda is a bank teller" (B) and that "Linda is a feminist and a bank teller" (F and B). Participants typically order the (F and B) event as more likely than the B event. There is an impressive amount of research replicating and extending this finding, which establishes its robustness (for a review, see Tentori, Crupi, & Russo, 2013). Of course, the conjunction fallacy does not occur all the time, and establishing when it does occur is a critical question. This question was recently addressed by Tentori et al. (2013), who put forth an argument that *inductive confirmation* (IC) rather than *perceived probability* (PP), described below, is a key determinant. Tentori et al. (2013) provided strong empirical support for this conclusion; moreover, they used this conclusion to rule out many previous formal explanations that relied on probabilistic dependence as the key factor. However, they went further and argued strongly that the *quantum probability* (QP) model (Busemeyer et al., 2011) was inconsistent with their empirical findings (see p. 239 and p. 247 in Tentori et al., 2013). Based on the results of Tentori et al. (2013), they repeated this point even more strongly in a subsequent publication (Tentori & Crupi, 2013). The purposes of this comment are twofold: (a) to clearly prove that our QP model is consistent with the empirical results of TCR, and (b) to describe experimental tests that can distinguish IC and QP theory by examining their a priori predictions.

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Simple Quantum Model of the Basic Findings

Perceived Probability Versus Inductive Confirmation

Consider the Linda problem again with B representing bank teller, F representing feminism, and L representing the Linda story. The notation $J(B \mid L)$ denotes the judged probability that Linda is a bank teller after being told the Linda story; $J(F \mid L)$ denotes the judged probability that Linda is a feminist after being told the Linda story; and $J(F \text{ and } B \mid L)$ denotes the judged probability that Linda is a feminist and a bank teller after being told the Linda story. The PP for a Hypothesis F is measured by first telling participants the Linda story and also telling them that she is a bank teller, and then asking participants to judge the probability that Linda is a feminist, which is denoted as $J(F \mid L \text{ and } B)$. The IC for Hypothesis F is measured by first telling participants the Linda story and also telling them that she is a bank teller, and then asking participants to judge the degree to which the feminism hypothesis is confirmed (positive) or disconfirmed (negative) by the Linda story, which is denoted as $c(F, L \mid B)$. Tentori et al. (2013) assumed that the sign of $c(F, L \mid B)$ is determined by the sign of the difference $J(F \mid L \text{ and } B) - J(F \mid B)$.[1] The conjunction fallacy occurs when $J(F \text{ and } B \mid L)$ exceeds $J(B \mid L)$. According to the PP account, this fallacy occurs because the PP of the feminism Hypothesis F is high; according to the IC account, this occurs because the IC of F is positive.

Both the PP and the IC accounts can explain the conjunction fallacy that occurs with the Linda problem because PP is high in this case, and IC is also positive (see Tentori et al., 2013). TCR designed experiments using new stories and hypotheses that distinguished these two accounts as follows. Define e as the evidence provided by some story, and define H_i as a hypothesis about the story. The basic design of Tentori et al. (2013) (see p. 241) is to compare the rate of conjunction fallacy when a hypothesis H_1 is combined with one of two other hypotheses: H_2 and H_3, where H_2 has a higher IC while H_3 has a higher PP. The IC account is empirically supported over the PP account if the following pattern occurs (see p. 241 and p. 247, Tentori et al., 2013): $J(H_3 \mid e \text{ and } H_1) > J(H_2 \mid e \text{ and } H_1)$, but $c(H_2, e \mid H_1) > c(H_3, e \mid H_1)$, and ($H_2$ and H_1) is chosen more frequently than (H_3 and H_1) as most likely to be true.

The Linda problem is an example of what is called the *M-A paradigm*, which provides explicit evidence e in the form of a story before making the judgments. Another paradigm is called the *A-B paradigm*, which does not provide any explicit evidence. For example, participants can be asked to judge the probability of randomly sampling a person from a health survey who is over 50 years old (Hypothesis H_1) and who has had a heart attack (Hypothesis H_2), and this is compared to the probability of randomly sampling a person from a health survey who has had a heart attack. The conjunction fallacy occurs when $J(H_1 \text{ and } H_2) > J(H_2)$. According to PP, this fallacy occurs when $J(H_2 \mid H_1)$ is high; according to IC, this fallacy occurs when $J(H_2 \mid H_1)$ exceeds $J(H_2)$.

TCR started their article with a compelling thought experiment, called the "black shoes" example, which used e = Linda story, H_1 = bank teller, H_2 = feminist, and H_3 = black shoes. Experiments 1 and 2 used the M-A paradigm (evidence was provided). Specifically, Experiment 1 used e = a Russian woman, H_1 = a New York inhabitant, H_2 = an interpreter, and H_3 = not an interpreter; Experiment 2 used e = a degree in violin, H_1 = a mountain climber, H_2 = a music lesson teacher, and H_3 = owns an umbrella. Experiments 3 and 4 used the A-B paradigm with e = no evidence (presumably sampling a person in Europe; the participants were Italian students in the Tentori et al., 2013, experiments), H_1 = an American (presumably from the U.S.), H_2 = overweight, and H_3 = owns an umbrella. These examples also vary the size of the PP, and so they provide a broad range of tests. The first part of this comment applies the QP model to these four prototypic examples from Tentori et al. (2013). Although the QP model generates predictions for all of the probabilities shown in Table 1, the data reported by Tentori et al. (2013) only included three statistics: (a) the average rating of PP for each hypothesis, (b) the average rating of IC for each hypothesis, and (c) the relative frequency that a conjunction error occurred.

Black Shoes and Other Examples

A compelling reason to argue against PP and in favor of IC is made by the following thought experiment (p. 236, Tentori et al., 2013). Suppose B represents the feature bank teller, F represents the feature feminist, S represents the owning black shoes feature, and L represents the evidence provided by the Linda story. Tentori et al. (2013) argued that the expected result for this case was that $J(B \text{ and } S \mid L) < J(B \text{ and } F \mid L)$. This pattern is contrary to PP because it is expected that $J(S \mid B, L) > J(F \mid B, L)$; it is consistent with IC because it is expected that $c(F, L \mid B) > c(S, L \mid B)$. The latter is based on the intuition that, because almost all women own black shoes, the Linda story does not produce any increase in the likelihood of owning black shoes, so the right-hand side is zero.

Now consider a simple QP model for this case (for a general introduction to QP theory, see Busemeyer et al., 2011 and Busemeyer & Bruza, 2012). The reader will notice that we have to make more assumptions than the IC hypothesis to account for findings presented in Tentori et al. (2013). There are two good reasons for this. First, QP theory generates *quantitative* values for all of the relevant probabilities, whereas the IC account only makes *qualitative* predictions for the co-occurrence of conjunction fallacies with positive IC. Second, the paradigm used by TCR was designed to directly test the IC hypothesis, which was not ideal for deriving a priori tests of QP theory. In the Concluding Comments section, we briefly present paradigms that do provide a priori tests of QP theory, but the main goal of this comment is to show that, contrary to the claims of Tentori et al. (2013), QP theory is consistent with their findings. It is also important to note that the basic setup and assumptions used in the first example are reused in all of the four examples that we consider in this comment. That is, the same principles are applied uniformly across all four examples. Furthermore, these same principles are used to account for many other phenomena, not covered by the IC hypothesis, such as conjunction fallacies with more than two events, disjunction fallacies, unpacking effects, and order effects on inference (Busemeyer et al., 2011).

In general, a person's state of beliefs about the presence or absence of various feature combinations is represented by a (unit length) vector in an N-dimensional space. For simplicity, we limited the following applications to a 4-dimensional space. Initially, we described this space using what we called the *occupation basis* (because it involves information about the bank teller occupation), which is defined by four axes, or more technically, four

[1] Assuming standard probability rules, $p(H_2 \mid e \wedge H_1) > p(H_2 \mid H_1) \rightarrow \frac{p(H_2 \mid H_1)p(e \mid H_2 \wedge H_1)}{p(e \mid H_1)} > p(H_2 \mid H_1)$, and dividing both sides by $p(H_2 \mid H_1)$, we obtain $p(e \mid H_2 \wedge H_1) > p(e \mid H_1)$. Thus, $c(H_2, e \mid H_1) > 0$ if $p(e \mid H_2 \wedge H_1) > p(e \mid H_1)$.

Table 1
Probabilities Computed From Quantum Probability Model for Four Examples

Black shoes	Violin (Exp. 2)	American (Exp. 4)	Russian (Exp. 1)
$p(S \mid L) = .97$	$p(U \mid C) = 1.0$	$p(U) = 1.0$	$p(\bar{I} \mid N) = .95$
$p(S \mid B, L) = .80$	$p(U \mid C, V) = .80$	$p(U \mid A) = .94$	$p(\bar{I} \mid N, R) = .55$
$p(F \mid L) = .84$	$p(L \mid C) = .06$	$p(O) = .16$	$p(I \mid N) = .05$
$p(F \mid B, L) = .28$	$p(L \mid C, V) = .09$	$p(O \mid A) = .85$	$p(I \mid N, R) = .45$
$p(B \mid L) = .07$	$p(C \mid V) = .05$	$p(A) = .011$	$p(N \mid R) = .05$
$p(S, B \mid L) = .06$	$p(U, C \mid V) = .04$	$p(U, A) = .01$	$p(N, \bar{I} \mid R) = .06$
$p(F, B \mid L) = .29$	$p(L, C \mid V) = .06$	$p(O, A) = .15$	$p(N, I \mid R) = .20$

Note. Exp. = Experiment. Column 1: B (H_1, bank teller), F(H_2, feminist), S (H_3, black shoes), L (e, Linda). Column 2: C (H_1, mountain climber), L (H_2, music lessons), U (H_3, owns umbrella), V (e, violin student). Column 3: A (H_1, American), O (H_2, overweight), U (H_3, owns umbrella). Column 4: N (H_1, New York), I (H_2, interpreter), $\bar{I}$(H_3, not interpreter), R (e, Russian woman).

basis vectors, that span the space.[2] These four basis vectors are symbolized by $[SB, S\bar{B}, \bar{S}B, \overline{SB}]$, where SB stands for the presence of feature combination S and B, and $S\bar{B}$ stands for the presence of feature combination S and $\bar{B}$, and so forth; given the Linda story, the person has beliefs about the presence of each of these four feature combinations. Technically, the strengths of these beliefs are quantified by the coordinates (also called amplitudes) assigned to the four basis vectors. For example, we used the coordinate vector $\alpha_L = [.239, .9562, .1195, .1195]^T$ to represent the beliefs from the Linda story when described in terms of the occupation basis (the numerical precision comes from normalizing the length of four integers). Note that the largest amplitude (.9562) is assigned to $S\bar{B}$ (consistent with the Linda story). This is just one example of many possible coordinates that account for the results, and many variations around this prototype also work.

An important property of the occupation basis is that beliefs about black shoes and bank tellers are represented by coordinates using the same basis vectors. By doing this, we made an important assumption, which is called the compatibility assumption in QP theory. We assumed that, when evaluating shoe features and the occupation of bank tellers, the order of evaluation does not matter, so that people can form beliefs about conjunctions of these two features. We argued that this makes sense for these two features because it is common knowledge that women have black shoes, so people have considerable experience with shoes and occupations and their joint characteristics are well-known, and one feature does not affect the meaning of the other.

What about feminism? In this case, we assumed that a person does not use a compatible representation based on all eight conjunctions formed by combining the binary values of all three features (e.g., S and B and F). This is plausible for several reasons. Maybe people lack sufficient experience with combinations of feminist attitudes simultaneously with the other two features to form a complete joint space of all three features. Indeed, it has been shown that increased experience with conjunctions reduces the rate of conjunction fallacies (Nilsson et al., 2013). Alternatively, it may require too much mental capacity or effort to form the 8-dimensional space required to represent the conjunctions of all three features. Instead, we assumed that a person evaluates some of the concepts serially, one at a time, using a lower dimensional representation. This does not mean that people cannot form judgments about pairs of concepts such as feminism and occupations; instead, this means that the judgment about these pairs of concepts needs to be performed serially in an order-dependent way. In fact, order effects are observed with the conjunction fallacy (Stolarz-Fantino et al., 2003). This key assumption that people fail to form joint representations of all events is consistent with previous explanations for the conjunction fallacy (Agnoli & Krantz, 1989; Nilsson, 2008; Wolfe & Reyna, 2009; Yamagishi, 2003).

To answer questions about feminism, QP theory assumes that a person relies on a different basis from what is used for occupations—that is, a different set of features which are related to feminism and other ideologies. In the QP model, another basis describing new features can be formed by *rotating* the occupation basis. So, when answering questions about feminism, we assume that the person rotates from the occupation basis to an *ideological basis* that contains feminism. We interpret the four rotated basis vectors as $[F, A, B, C]$, where F is a feminist-type woman and A, B, C are three other types of ideologies (other than feminism). According to QP theory, we assumed that the occupation basis, used to describe bank tellers, is different from, technically *incompatible* with, the ideological basis used to describe feminism.[3] Because no joint representation of occupations and ideologies is manageable, questions concerning them have to be answered serially (rotating from one to the other) and the order of questioning matters.

This leaves us with the important issue of how to rotate from the occupation basis to the ideology basis. This is the key (and technically difficult) part of quantum theory (see Busemeyer & Bruza, 2012 for details). Here we use perhaps the simplest rotation. Consider the following 2×2 rotation matrix for rotating (coun-

[2] Technically, we should call this the tensor product Shoes $\times$ Bank Teller basis, but this name is too long, and so we will just refer to the shorter name.

[3] The words "compatible" and "incompatible" are technical terms in QP theory, and they should not be confused with their natural language usage. Incompatible does not mean mutually exclusive or orthogonal (on the contrary, mutually exclusive events are always compatible in this technical sense). Instead, being incompatible means the events are noncommutative, and cannot be defined simultaneously using the same basis.

terclockwise) two orthogonal axes by an angle θ_i within a 2-dimensional space

$$U(\theta_i) = \begin{bmatrix} \cos(\pi \cdot \theta_i) & \sin(\pi \cdot \theta_i) \\ -\sin(\pi \cdot \theta_i) & \cos(\pi \cdot \theta_i) \end{bmatrix}.$$

Alternatively, the angle θ_i is used to rotate (clockwise) the coordinates that describe the belief state. The angle (measured in radians) $-.5 \leq \theta_i \leq .5$ determines how much to rotate the bases (negative angles rotate in the opposite direction of positive angles). Setting $\theta_i = 0$ produces no rotation (leaving the coordinates the same), increasing θ_i increases the degree of change in coordinates until $\theta_i = .5$ completely reverses the coordinates (e.g., if the first dimension were certain to be true in one basis, then it becomes certain to be false in the other basis). Setting $\theta_1 = .25$ has the following effect: If you were certain about the first dimension in the first basis, then you would assign equal likelihood to either dimension in the rotated basis.

This rotation matrix can be extended to a 4-dimensional space by combining two such rotations to form a 4-dimensional rotation matrix, denoted as $U(\theta_1, \theta_2)$.[4] Consider rotating the four coordinates of the occupation basis to the four coordinates for the ideology basis. First, θ_1 is used to rotate each 2-dimensional subspace for the presence or absence of bank teller—it is used to rotate the pair of coordinates $(SB, S\overline{B})$ when black shoes are present, and it is also used to rotate the pair of coordinates $(\overline{S}B, \overline{SB})$ when black shoes are absent. Second, θ_2 is used to rotate each 2-dimensional subspace for presence or absence of black shoes—it is used to rotate the pair of coordinates $(SB, \overline{S}B)$ when bank teller is present, and it is also used to rotate the pair of coordinates $(S\overline{B}, \overline{SB})$ when bank teller is absent. For this first example, we simply used $\theta_1 = .30$ and $\theta_2 = 0$, which only rotates the coordinates for presence or absence of bank teller, because the other rotation for black shoes is not needed for this particular example.

Recall that the coordinates for the occupation basis were previously defined as $\alpha_L = [.239, .9562, .1195, .1195]^T$. Using the rotation matrix, $U(\theta_1, \theta_2)$, we can obtain the coordinates for the ideology basis from the coordinates in the occupation basis: The state vector, produced by the same story, but now expressed in the ideology basis, is given by the matrix product $\beta_L = U(.30, 0) \cdot \alpha_L = [.9141, .3686, .1669, -.0264]^T$. Note that using this rotated basis, the largest amplitude (.9141) is now assigned to the first coordinate representing the feminism (F) type (consistent with the Linda story).[5] Furthermore, we can rotate from the ideology basis back to the occupation basis by the inverse rotation $\alpha_L = U^{-1} \cdot \beta_L$. Thus we can start with either basis and rotate to the other. The power of quantum reasoning is that it allows a person to evaluate the same state from different perspectives (technically, different bases, using different coordinate systems).

According to the QP model, the probability of answering "yes" to a question is determined by matching the person's beliefs to the features corresponding to the question. Technically, this match is performed by projecting the state vector onto a subspace representing the answer to a question, and then taking its squared magnitude. This projection is very easy to do when working with the appropriate coordinates. For example, when using the occupation basis, we define a projector for the answer "yes" to the "shoes" question as $M_S = \mathrm{diag}[1, 1, 0, 0]$ which simply picks out the first two coordinates of α_L, that is, the projection is $M_S\alpha_L = [.239, .9562, 0, 0]^T$. Also when using the occupation basis, we define the projector for the answer "yes" to the bank teller question as $M_B = \mathrm{diag}[1, 0, 1, 0]$, which simply picks out the first and third coordinates of α_L. When using the ideology basis, we define the projector for the answer "yes" to the feminism question as $M_F = [1, 0, 0, 0]$, which simply picks out the first coordinate of β_L, that is, the projection is $M_F\beta_L = [.9141, 0, 0, 0]^T$.

Finally, the squared length of the final projection equals the probability of an answer or series of answers. The probability of "yes" to the bank teller question equals $p(B \mid L) = \| M_B\alpha_L \|^2 = 0.239^2 + 0.1195^2 = .0714$. The probability of "yes" to the feminist question equals $p(F \mid L) = \| M_F\beta_L\|^2 = .9141^2 = .8356$. The probability of "yes" to the black shoes question and "yes" to the bank teller question is determined by first projecting on black shoes, and then projecting on bank teller, which equals $p(S, B \mid L) = p(S \mid L) \cdot p(B \mid S, L) = \| M_B \cdot M_S \cdot \alpha_L \|^2 = .239^2 = .0571$. The probability of "yes" to the feminist question and then "yes" to the bank teller question is obtained by first projecting on feminism, then rotating to the occupation basis, and finally projecting on bank teller, which equals $p(F, B \mid L) = p(F \mid L)p(B \mid F, L) = \| M_BU^TM_F\beta_L \|^2 = .2887$. Note that the latter result reproduces the conjunction fallacy, because $p(F, B \mid L) = .2887 > .0714 = p(B \mid L)$. This probability depends on feminism being evaluated first, and the order of the two judgments matters. We generally assume that the more likely event, in this case feminism, is evaluated before the less likely event, in this case, bank teller (see Busemeyer et al., 2011). However, if the person is first informed that Linda is in fact a bank teller, then the participant can also compute the probability of feminist, conditioned on "yes" to bank teller, as follows $p(F \mid B,L) = \dfrac{\| M_FUM_B\alpha_L \|^2}{p(B \mid L)} = \dfrac{.0197}{.07} = .2764$. (See Busemeyer & Bruza, 2012 for a complete review of the quantum axioms).

We summarized the calculations from the QP model for the black shoes example in Table 1, column 1, for which we obtain $p(S \mid B, L) > p(F \mid B, L)$, but $p(F \mid B, L) > p(B \mid L)$ and $p(F, B \mid L) > p(B \mid L) > p(S, B \mid L)$. Table 1 also shows the probabilities computed for the other three examples used in the four experiments reported in Tentori et al. (2013). The other three examples are treated using exactly the same rules as described above (e.g., 4d space the same type of rotation, the same types of projections, but different bases representing different types of features, and different state vectors representing different background stories). These details are presented in the appendix. (The MATLAB computer programs used to compute all probabilities are available upon request). All of the probabilities in the last three columns of Table 1 are in ordinal agreement with all of the reported results in Tentori et al. (2013).[6]

Unfortunately, it is not possible to determine how well the probabilities in Table 1 fit quantitatively because these probability judgments were not empirically observed by Tentori et al. (2013). The parameters that we chose are therefore somewhat arbitrary,

[4] Technically, this is done by using a Kronecker product $U = U(\theta_2, \theta_1) = U(\theta_2) \times U(\theta_1)$.

[5] Amplitudes do not have to be positive numbers because the probabilities are eventually obtained by their squared magnitude, as described next.

[6] In particular, for the last three columns, the second row exceeds the fourth row, the first row exceeds the second row, the fourth row exceeds the third row, and the last row exceeds the sixth row.

and they are only used to show that QP theory is *not disproven* by the TCR results. If quantitative empirical results for Table 1 become available, then we can more rigorously test the fit of the quantum model.

Summary

In sum, the QP probabilities presented in Table 1 are *consistent* with all of the main findings reported by Tentori et al. (2013). These probabilities provide counter examples to the claim that the QP model is inconsistent and falsified by the Tentori et al. (2013) findings. The reason the arguments presented in Tentori and Crupi (2013) cannot be used to disconfirm our QP model is that their analysis was restricted to a 2-dimensional space. In fact, we clearly stated (Busemeyer et al., 2011) that a realistic model requires a high-dimensional space (much greater than 2) to accommodate all the types of questions that we can ask a person. (A 2-dimensional space is only used as a toy example for illustration of the basic ideas.) The four examples presented above use a 4d space, which is sufficient for accounting for the main empirical findings reported by Tentori et al. (2013), but not necessarily realistic either.[7] High-dimensional spaces are commonly used in cognitive models of probability judgment (cf. Dougherty et al., 1999).

Why does the QP model work well for explaining the findings from this conjunction-fallacy paradigm? The essential reason in all four examples is that (a) the extremely common feature (e.g., owning black shoes) is compatible with one of the hypotheses (e.g., bank teller); (b) the extremely common feature generates such a high probability that additional evidence does not add anything; and (c) one of the hypotheses (e.g., bank teller) is incompatible with another hypothesis (e.g., feminism). On the one hand, the compatibility with the extremely common feature prevents the conjunction fallacy from occurring when the extremely common feature is involved; on the other hand, the incompatibility between the other two hypotheses produces a conjunction error. This is the way that the QP model reproduces the observed pattern of results reported in Tentori et al. (2013).

Empirically Distinguishing the Quantum Versus Confirmation Accounts

Both the quantum and confirmation accounts of the conjunction fallacy depend on the presence of a critical antecedent condition. Our QP model required the two events to be incompatible, and we needed to first empirically determine compatibility or incompatibility (e.g., by testing for order effects of the two events). The IC account required the confirmation to be positive, and Tentori et al. (2013) needed to first empirically determine positivity or negativity (by obtaining confirmation-strength judgments). In addition, both the quantum and confirmation accounts of the conjunction fallacy are asymmetric with respect to the two hypotheses, H_1, and H_2. According to the QP model, if the events are incompatible, then the projections are noncommutative, $p(H_1|e)p(H_2|H_1,e) \neq p(H_2|e) p(H_1|H_2,e)$, and we assume that the more likely marginal event is evaluated first. According to the IC account, the measurement of confirmation $c(H_2, e \mid H_1)$ is not necessarily the same as $c(H_1, e \mid H_2)$, and Tentori et al. (2013) argued that it seemed more relevant to evaluate the confirmation for the added conjunct H_2 when comparing H_1 and H_2 with H_1 alone. However, this asymmetry works quite differently for the two models, which leads to two interesting empirical tests to distinguish them.

First, consider the Linda problem once again, but suppose that we manipulate the order of questions. For both orders, the participant is first told the Linda story. For Order 1, the participant is first asked to judge the probability of (F and B) in isolation (not knowing whether any other question comes next). Afterward, the participant is asked to judge the probability of B. For Order 2, the participant is asked to judge the probability of B first, and then the probability of (F and B).

According to the QP model, the conjunction fallacy is predicted to occur more frequently for Order 1 than for Order 2 (see Busemeyer et al., 2011). Using Order 1, we assumed that the person would compute $p(F \mid L)p(B \mid F, L)$ for the first question, and then $p(B \mid L)$ for the second, and in this order, $p(F \mid L)p(B \mid F, L) > p(B \mid L)$. Using Order 2, the person would first compute $p(B \mid L)$, and having this answer in hand, was then encouraged to compute the second question using $p(B \mid L)p(F \mid B, L)$, and in this order, the QP model must predict $p(B \mid L)p(F \mid B, L) < p(B, L)$ because $p(F \mid B, L) < 1$).

According to the IC account, the conjunction fallacy is clearly predicted to occur for Order 2 but not necessarily for Order 1 for the following reason. Using Order 2, the person is asked to judge the likelihood of bank teller occupation, given the Linda story, and having the background Hypothesis B in hand, the second question would introduce an added Hypothesis F, so now the person would consider confirmation of F, conditioned on the background of B, producing $c(F, L \mid B)$, which is positive, and so the conjunction fallacy is predicted to occur. Using Order 1, the person is asked to judge (F and B) in isolation (not knowing whether any other question comes next), and according to Tentori (personal communication, 2014), the person uses $c(F \text{ and } B, L)$ to evaluate $J(F \text{ and } B \mid L)$ and the sign of $c(F \text{ and } B, L)$ can be positive or negative.

We found it interesting that this experiment actually has been conducted (see Stolarz-Fantino et al., 2003, Experiment 2), and the results were that the conjunction fallacy occurred with Order 1 and not with Order 2, which agreed with the prediction of the QP model. Gavanski and Roskos-Ewoldsen (1991) also examined two different orders and found a similar pattern of results. There may be many reasons for order effects, but they do modify the occurrence of the conjunction errors, and so a theory that accounts for this moderating effect is clearly preferred to another that does not.

A second test of the quantum versus the confirmation account can be achieved by directly manipulating compatibility. The IC account of the conjunction fallacy only depends on a positive confirmation $c(H_2,$

[7] A more realistic model would have to allow, e.g., for many different types of occupations other than bank teller. Also, a more realistic model could allow shoes to be compatible with the ideology basis too, which can be done, but would require more than 4 dimensions. We chose not to use a higher dimensional model because higher dimensions were not needed to reproduce the Tentori et al. (2013) results.

$e \mid H_1) > 0$. According to QP theory, the conjunction fallacy depends on an incompatible representation of events, which may be changed into a compatible representation by presenting the events in two-way tables or nested sets (Busemeyer et al., 2011). Joint representations of events would encourage *use of a single compatible basis involving all combinations*. Assuming that judged confirmation does not change with manipulations of compatibility, then this manipulation can be used to discriminate between the two accounts.

In fact, experiments manipulating representation to encourage usage of joint representations have been highly effective at eliminating conjunction errors (Agnoli & Krantz, 1989; Nilsson, 2008; Wolfe & Reyna, 2009; Yamagishi, 2003; Nilsson et al., 2013). These results suggest that failure to form a joint representation (which corresponds to incompatibility in QP theory) is the primary source of the conjunction fallacy (Sloman et al., 2003; Reyna & Brainerd, 2008).

Concluding Comments

The application of QP theory to human judgment and decisions is new, and new ideas are rightfully questioned and demand more evidence than usual. What strong a priori predictions does the QP model make regarding probability judgments? We have already described many in detail (see Busemeyer et al., 2011), but it is useful to summarize a few of the predictions here. One could argue that our account of the conjunction fallacy was somewhat post hoc, because we did not make an a priori prediction that feminism and bank teller were incompatible events. However, once we made this assumption, we had to predict order effects, and these effects had, in fact, been obtained (Stolarz-Fantino et al., 2003; Gavanski & Roskos-Ewoldsen, 1991). Furthermore, once we made this assumption, then a number of *other* predictions had to follow a priori (for *any* dimension N, choice of rotation, and state vector). First of all, our QP model predicted disjunction errors, $p(B \mid L) < p(F \text{ or } B \mid L) < p(F \mid L)$, for the same events. This is because the disjunctive probability equaled 1 – the probability of the conjunction ($\overline{B}$ *and* $\overline{F}$), and the latter was predicted to produce a conjunction error because of incompatibility. Indeed, it has been found that disjunction errors are also obtained using the same events that produce conjunction errors (Morier & Borgida, 1984; Fisk, 2002; Yates & Carlson, 1986). Furthermore, another directly testable prediction of the QP model concerns conditional probabilities: The QP model must predict that $p(B \mid F, L) \geq p(B \mid L)$, because the QP model for the conjunction fallacy implies that $p(B \mid L) < p(F \mid L) \times p(B \mid F, L) \leq p(B \mid F, L)$. This prediction also has been supported by past research (Fisk & Pidgeon, 1998). The QP model allows both conjuncts to be judged higher than the conjunction, or the conjunction can be judged higher than one of the conjuncts, but it does not allow the conjunction to be judged higher than both conjuncts. Empirically, conjunction errors occur most frequently when the conjunction is judged in between the two conjuncts (Gavanski & Roskos-Ewoldsen, 1991).

The strongest prediction made to date by our QP model concerns order effects for binary (e.g., yes, no) judgments about pairs of events. According to the QP model, if two events are incompatible, we must predict order effects when deciding about the pair of events, for example, $p(Ay \text{ and then } Bn) \neq p(Bn \text{ and then } Ay)$, where for example $p(Ay \text{ and then } Bn)$ is the probability of saying "yes" to question A and then "no" to question B and $p(Bn \text{ and then } Ay)$ is the probability of saying "no" to question B and then "yes" to question A. But much more important than that, the QP model must predict a very special pattern of order effects! According to the QP model (just as a reminder—for any dimension N, rotation, and initial state), the pattern of order effects must satisfy an exact, empirically observable constraint that we call the QQ equality (see Wang & Busemeyer, 2013): $p(Ay \text{ and then } Bn) + p(An \text{ and then } By) = p(Bn \text{ and then } Ay) + p(By \text{ and then } An)$. This is an a priori, *precise*, *quantitative*, and *parameter-free* prediction about the pattern of order effects, and it has been statistically supported across a wide range of 70 national field experiments (containing 651 to 3,006 nationally representative participants per field experiment) that examined question-order effects (Wang et al., 2014).

The goals of QP theory are different from the IC hypothesis. The goal of QP theory is to provide a coherent theory for any kind of probability judgment, such as conjunctions and disjunctions of two or more events (Busemeyer et al., 2011), and hypotheses conditioned on one piece of evidence or more presented in different orders (Trueblood & Busemeyer, 2010). In contrast, the IC has a more restrictive goal, which is to identify the primary determinant of conjunction fallacies for two conjuncts. The main point of this comment is that there is no inherent inconsistency between QP theory and the importance of IC as a determinant of the conjunction fallacy. Instead, if IC is critical, then this determinant imposes constraints that QP theory must satisfy. The added value of QP theory is to make predictions for additional factors, such as order effects or training with conjunctions, that moderate the conjunction fallacy, and to make predictions for other probability judgment errors, such as the closely related disjunction fallacy.

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Appendix

Fits to the Experimental Data of Tentori et al. (2013)

The Violin Example

The "violin" example, used in Experiment 2 of Tentori et al. (2013), is treated in a similar manner to the "black shoes" example. The judge is asked to consider a person with a violin degree ($e = V$). Then three hypotheses are considered: The first is C representing "this person is a mountain Climber;" the second is L representing "this person teaches music Lessons;" and the third is U representing "this person owns an Umbrella." To determine the IC for each hypothesis, we need to define a state before the evidence V is presented, and again after the evidence V is presented. For this example, we define a *Climber basis* that has four basis vectors: $[UC, U\overline{C}, \overline{U}C, \overline{UC}]$, where for example, UC represents (U *and* C) both present, $U\overline{C}$ represent (U *and* $\overline{C}$), etc. With respect to this climber basis, we use the coordinate vector $\alpha_0 = [.1104, .9938, 0, 0]$ before the evidence, and we use the coordinate vector $\alpha_V = [0.1952, 0.9759, 0.0976, 0]$ after the evidence. The coordinate vector α_V is obtained by rotating α_0 to a basis for violin, projecting on violin, and then rotating back to the climber basis. In both states, the second coordinate representing $U\overline{C}$ has the largest amplitude; the violin evidence has the effect of diffusing and spreading out the amplitudes a bit. (Again this is one example, and variations around this example also reproduce the TCR findings). We define *Lesson basis* that uses the four basis vectors $[L, A, B, C]$ where the first coordinate represents the activity of music lessons, and the others represent three other lesson activities. (We can allow umbrella to be compatible with Lesson too, but as mentioned in Footnote 5, this requires a higher dimensionality, which we do not need to reproduce the results.) The unitary operator that rotates from the climber basis to the lesson basis is defined as $U(\theta_1 = .4, \theta_2 = .2)$, and this is used to compute the coordinates for the lessons basis $\beta = U \cdot \alpha$. (These rotation parameters provide one example, and many variations, such as ($\theta_1 = .2, \theta_2 = 0$), also reproduce the TCR findings). The projector for "yes" to C in the climber basis is $M_C = diag[1, 0, 1, 0]$; the projector for "yes" to U in the climber basis is $M_U = diag[1, 1, 0, 0]$; the projector for L in the lesson basis is $M_L = diag[1, 0, 0, 0]$. We summarize the calculations for this example in Table 1. These probabilities are ordinally consistent with all of the experimental results of Experiment 2 in Tentori et al. (2013).

The American Example

The Linda and Violin problems are examples of what is called the M-A paradigm, where evidence, that is a story, is provided. The next example (prototypical of Experiments 3 and 4 in Tentori et al. (2013)) uses what is called the A-B paradigm, where no obvious evidence is provided. Three hypotheses are considered: The first is A representing "is an American;" the second is O representing "is overweight;" and the third is U representing "owns an umbrella." For this example, we define an *American basis* that has four basis vectors: $[UA, U\overline{A}, \overline{U}A, \overline{UA}]$, and the state vector is assigned coordinates $\alpha = [.0995, .9947, .0249, 0]$. (This presumably reflects the Italian participants' background knowledge of the prevalence of US Americans in Europe). Once again, the largest amplitude is assigned to the second coordinate representing $U\overline{A}$. We define an *overweight basis* that uses the four basis vectors $[O, A, B, C]$ where the first coordinate represents overweight, and the others represent other weight categories. The unitary operator that rotates from the American basis to the overweight basis is defined as $U(\theta_1 = .1, \theta_2 = 0)$ (and variations around this give similar results). This is used to compute the coordinates for the overweight basis $\beta = U \cdot \alpha$. The projector for "yes" to A in the American basis is $M_A = diag[1, 0, 1, 0]$; the projector for "yes" to U

(*Appendix continues*)

in the American basis is $M_U = diag[1, 1, 0, 0]$; the projector for O in the overweight basis is $M_O = diag[1, 0, 0, 0]$. We summarize the calculations for this example in Table 1. These probabilities are again ordinally consistent with all of the experimental results of Experiments 3 and 4 in Tentori et al. (2013).

The Russian Women Example

The last example comes from the first experiment in Tentori et al. (2013), which used the M-A paradigm, but it was different from the Violin example because it used the negation of one hypothesis as another hypothesis. Initially the judge is asked to consider a woman from New York, and later the judge is told that this NY women is Russian ($e = R$). Three hypotheses are considered: The first is N representing "this person is a woman from New York;" the second is I representing "this person is an interpreter;" and the third is $\bar{I}$ representing "this person is not an interpreter." To determine the IC for each hypothesis, we need to define a state before the evidence R is presented, and again after the evidence R is presented. For this example, we define a *New York basis* that has four basis vectors: $[NX, N\bar{X}, \bar{N}X, \overline{NX}]$, where for example NX represents (N *and* X) and X is some other feature related to New York. With respect to the New York basis, we use the coordinate vector $\alpha_N = [0.8944, 0.4472, 0, 0]$ when the woman is described as being from New York; and we used the coordinate vector $\alpha_R = [-0.1952, 0.0976, 0, 0.9759]$ when the woman is described as being a Russian. Both of these are obtained by projecting some other initial state (α_0, before either New York or Russian is known) onto the subspace for either a New York women or for a Russian woman, and then expressing this state in the New York basis. We define an *occupation basis* that uses the four basis vectors $[A, B, I, C]$ where the third coordinate represents the interpreter occupation, and the others (event $\bar{I}$, that is not I) represent three other occupations that are not interpreters. Note that in this example, the event I is compatible (in the technical quantum sense) with the mutually exclusive event $\bar{I}$ (i.e., they are both represented using the same occupation basis). The unitary operator that rotates from the occupation basis to the New York basis is defined as $U(\theta_1 = .25, \theta_2 = .25)$, which is used to compute the coordinates for the New York basis from the occupation basis as follows: $\beta = U \cdot \alpha$; the rotation from the New York basis to occupation basis is U^T, which is used to compute the coordinates for the occupation basis from the New York basis as follows: $\alpha = U^T \cdot \beta$. The projector for "yes" to N in the New York basis is $M_N = diag[1, 1, 0, 0]$; the projector for "yes" to I in the occupation basis is $M_I = diag[0, 0, 1, 0]$; the projector for $\bar{I}$ in the occupation basis is $M_{\bar{I}} = diag[1,1,0,1]$. We summarize the calculations for this example in Table 1. These probabilities are again ordinally consistent with all of the experimental results of Experiment 1 in Tentori et al. (2013).

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