Method of Substitution: Change of Variable

Theorem: *Indefinite Integral by Substitution.* Suppose that the function g has a continuous derivative and that f is continuous on the range of g. Let $u = g(x)$. Then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example: Find

$$\int x^2\sqrt{x^3+9}dx.$$

Solution: Note that x^2 is, to within a constant factor, the derivative of x^3+9. We can, therefore, substitute

$$u = x^3 + 9, \qquad du = 3x^2dx.$$

Thus,

$$\begin{aligned}\int x^2\sqrt{x^3+9}dx &= \int x^2\sqrt{x^3+9}\tfrac{1}{3}\cdot 3dx\\ &= \tfrac{1}{3}\int x^2\sqrt{x^3+9}3dx\\ &= \tfrac{1}{3}\int\sqrt{u}du = \tfrac{1}{3}\int u^{1/2}du\\ &= \tfrac{1}{3}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{2}{9}u^{3/2} + C\\ &= \tfrac{2}{9}(x^3+9)^{3/2} + C.\end{aligned}$$

Theorem: *Definite Integral by Substitution.* Suppose that the function g has a continuous derivative on $[a,b]$ and that f is continuous on the set $g([a,b])$. Let $u = g(x)$. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example: Evaluate

$$\int_0^{\sqrt{\pi}} 3x\cos(x^2)dx$$

Solution: Note that x is, to within a constant factor, the derivative of $\cos(x^2)$. We can, therefore, substitute

$$u = \cos(x^2), \qquad du = 2xdx.$$

To change the limits of integration, we evaluate u at $x = 0$ and $x = \sqrt{\pi}$ to find

$$x = 0 \Longrightarrow u = 1, \qquad x = \sqrt{\pi} \Longrightarrow u = -1$$

Thus,

$$\begin{aligned}\int_0^{\sqrt{\pi}} 3x\cos(x^2)dx &= \tfrac{3}{2}\int_1^{-1}\cos u\,du\\ &= \tfrac{3}{2}\sin u\big|_1^{-1} = \tfrac{3}{2}(\sin(-1) - \sin 1) = -3\sin 1\end{aligned}$$